# Evaluating $\zeta(2m)$ via Telescoping Sums.

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# Outline

# Basel Problem

Find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$

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- Gained notoriety when Jakob Bernoulli wrote about it in 1689.
- Remained unsolved into the 1730's.

# Answer to the Basel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

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- First established by Leonhard Euler in 1734.
- Subsequently proved in many distinct ways (Multivariate change of variables from Calculus, Fourier Series, Complex Analysis, etc.)
- Daniel Daners published a proof of this fact using basic Calculus in 2012.

# Daner's proof for $\zeta(2)$ using telescoping sums

- Define the following families of integrals (with $n$ being a nonnegative integer):

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx, \; B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx.$$

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- Initial values: $A_0 = \frac{\pi}{2}$, and $B_0 = \frac{\pi^3}{24}$.
- Recurrence Relations for $n > 0$ (via integration by parts): $A_n = \dfrac{2n-1}{2n} A_{n-1}$ and $A_n = n(2n-1)B_{n-1} - 2n^2 B_n$.

# Daner's proof for $\zeta(2)$ using telescoping sums, continued

- Divide both sides of the second recurrence by $n^2 A_n$ and substitute the first recurrence:

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right).$$

- (Telescoping step) Sum both sides from $n = 1$ to $k$:

$$\sum_{n=1}^{k} \frac{1}{n^2} = 2\left(\frac{B_0}{A_0} - \frac{B_k}{A_k}\right) = \frac{\pi^2}{6} - 2\frac{B_k}{A_k}.$$

# Daner's proof for $\zeta(2)$ using telescoping sums, concluded

- Since $\sin x \geq \frac{2x}{\pi}$ on $[0, \frac{\pi}{2}]$, observe that

$$B_k = \int_0^{\frac{\pi}{2}} x^2 \cos^{2k} x \, dx \leq \int_0^{\frac{\pi}{2}} \left(\frac{2}{\pi} \sin x\right)^2 \cos^{2k} x \, dx = \frac{\pi^2 A_k}{8(k+1)}.$$

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- Then $0 < \dfrac{B_k}{A_k} \leq \dfrac{\pi^2}{8(k+1)}$, and so $\displaystyle\lim_{k\to\infty} \frac{B_k}{A_k} = 0$.

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- Then $0 < \dfrac{B_k}{A_k} \leq \dfrac{\pi^2}{8(k+1)}$, and so $\lim\limits_{k\to\infty} \dfrac{B_k}{A_k} = 0$.
- Finally, letting $k \to \infty$ yields $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. ■

# Beyond the Basel Problem

- More generally, define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

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- More generally, define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.
- Euler also proved that for any $m \in \mathbb{N}$

$$\zeta(2m) = \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{(-1)^{m+1} 2^{2m-1} B_{2m}}{(2m)!} \pi^{2m}$$

where $B_k$ denotes the $k$-th Bernoulli number.

First few: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$.
Moreover, $B_3 = B_5 = B_7 = ... = 0$.

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- $\zeta(12) = 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \cdots = \frac{691\pi^{12}}{638512875}$

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# Solving for the even zeta constants

- As with the Basel Problem, there are many distinct solutions (though not quite as many).
- For the remainder of this discussion, I will demonstrate how to extend Daner's telescoping sum technique to compute $\zeta(2m)$ for any positive integer $m$.
- As a warm-up for the general case, I will first use this technique to compute $\zeta(4)$.

# A proof for $\zeta(4)$ using telescoping sums

- Define $A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$, $B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx$, and $C_n = \int_0^{\frac{\pi}{2}} x^4 \cos^{2n} x \, dx$.

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- Initial values: $A_0 = \frac{\pi}{2}$, $B_0 = \frac{\pi^3}{24}$, $C_0 = \frac{\pi^5}{160}$.

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- Initial values: $A_0 = \frac{\pi}{2}$, $B_0 = \frac{\pi^3}{24}$, $C_0 = \frac{\pi^5}{160}$.
- Recurrence relations (for $n > 0$):
  $A_n = \frac{2n-1}{2n} A_{n-1}$
  $A_n = n(2n-1)B_{n-1} - 2n^2 B_n$
  $B_n = \frac{1}{6}\left(n(2n-1)C_{n-1} - 2n^2 C_n\right)$

# A proof for $\zeta(4)$ using telescoping sums, continued

- Divide both sides of the last two recurrences by $n^2 A_n$ and substitute the first recurrence:

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) \text{ and } \frac{1}{n^2}\frac{B_n}{A_n} = \frac{1}{3}\left(\frac{C_{n-1}}{A_{n-1}} - \frac{C_n}{A_n}\right)$$

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- Telescoping step - Sum the first equation from $n = 1$ to $k$ and the second from $k = 1$ to $N$ (after changing its index):

$$\sum_{n=1}^{k} \frac{1}{n^2} = 2\left(\frac{\pi^2}{12} - \frac{B_k}{A_k}\right) \text{ and } \sum_{k=1}^{N} \frac{1}{k^2}\frac{B_k}{A_k} = \frac{1}{3}\left(\frac{\pi^4}{80} - \frac{C_N}{A_N}\right)$$

# A proof for $\zeta(4)$ using telescoping sums, continued

- The first equation yields $\dfrac{B_k}{A_k} = \dfrac{\pi^2}{12} - \dfrac{1}{2}\displaystyle\sum_{n=1}^{k} \dfrac{1}{n^2}$.

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- The first equation yields $\dfrac{B_k}{A_k} = \dfrac{\pi^2}{12} - \dfrac{1}{2}\displaystyle\sum_{n=1}^{k}\dfrac{1}{n^2}$.
- Substitute this into the second equation:

$$\frac{\pi^2}{12}\sum_{k=1}^{N}\frac{1}{k^2} - \frac{1}{2}\sum_{k=1}^{N}\sum_{n=1}^{k}\frac{1}{k^2 n^2} = \frac{1}{3}\left(\frac{\pi^4}{80} - \frac{C_N}{A_N}\right).$$

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- Substitute this into the second equation:

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- Rewrite this to expose $\sum \frac{1}{k^4}$:

$$\frac{\pi^2}{12}\sum_{k=1}^{N}\frac{1}{k^2} - \frac{1}{2}\left(\sum_{k=1}^{N}\frac{1}{k^4} + \sum_{\substack{j<k \\ j,k=1}}^{N}\frac{1}{j^2k^2}\right) = \frac{1}{3}\left(\frac{\pi^4}{80} - \frac{C_N}{A_N}\right).$$

# A proof for $\zeta(4)$ using telescoping sums, continued

- However,

$$\Big(\sum_{k=1}^{N} \frac{1}{k^2}\Big)^2 = \sum_{k=1}^{N} \frac{1}{k^4} + 2 \cdot \sum_{\substack{j<k \\ j,k=1}}^{N} \frac{1}{j^2 k^2}.$$

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- Use this relation to eliminate the nested sum:

$$\frac{\pi^2}{12}\sum_{k=1}^{N}\frac{1}{k^2}-\frac{1}{4}\left[\sum_{k=1}^{N}\frac{1}{k^4}+\left(\sum_{k=1}^{N}\frac{1}{k^2}\right)^2\right]=\frac{1}{3}\left(\frac{\pi^4}{80}-\frac{C_N}{A_N}\right).$$

# A proof for $\zeta(4)$ using telescoping sums, concluded

- Let $N \to \infty$, noting that $\frac{C_N}{A_N} \to 0$ (as before):

$$\frac{\pi^2}{12}\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{4}\left[\sum_{k=1}^{\infty}\frac{1}{k^4} + \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right] = \frac{\pi^4}{240}.$$

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- Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$, substituting this into the previous relation and solving for the remaining sum gives $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$. ■

# Overview

- We now prove the general case:

$$\zeta(2m) = \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = \frac{(-1)^{m+1} 2^{2m-1} B_{2m}}{(2m)!} \pi^{2m}$$

- This proof is similar to the previous two cases, but with a couple more difficulties:
  - Evaluating the nested sum $S(k) = \sum_{1 \leq i_1 \leq \cdots \leq i_k < \infty} \frac{1}{i_1^2 \cdot \ldots \cdot i_k^2}$.
  - Using the Bernoulli numbers.

# Dealing with the nested sum

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Proof (sketch):

- Define $S(0) = 1$, and consider the generating function $G(x) = \sum_{k=0}^{\infty} S(k)x^{2k} = 1 + S(1)x^2 + S(2)x^4 + \cdots$.

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- $G(x) = \prod_{j=1}^{\infty} \frac{1}{1-\frac{x^2}{j^2}} = \frac{\pi x}{\sin \pi x}$, by the infinite product for sine.

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- $G(x) = \prod_{j=1}^{\infty} \frac{1}{1-\frac{x^2}{j^2}} = \frac{\pi x}{\sin \pi x}$, by the infinite product for sine.
- Next, $G(x) = 1 - 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^2}{x^2-n^2}$, by the infinite partial fraction expansion of cosecant.

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- Next, $G(x) = 1 - 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^2}{x^2-n^2}$, by the infinite partial fraction expansion of cosecant.
- Since $\frac{x^2}{x^2-n^2} = -\sum_{k=1}^{\infty} (\frac{x}{n})^{2k}$ by the geometric series, applying this yields $G(x) = 1 + 2 \cdot \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} x^{2k}$.

# Dealing with the nested sum, continued

Proof (sketch), continued:

- However, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{2}{(2n)^{2k}} = (1 - 2^{1-2k})\zeta(2k)$.

# Dealing with the nested sum, continued

Proof (sketch), continued:

- However, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{2}{(2n)^{2k}} = (1 - 2^{1-2k})\zeta(2k).$
- So, $G(x) = 1 + \sum_{k=1}^{\infty} 2(1 - 2^{1-2k})\zeta(2k)x^{2k}.$

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- So, $G(x) = 1 + \sum_{k=1}^{\infty} 2(1 - 2^{1-2k})\zeta(2k)x^{2k}$.
- Since $G(x) = \sum_{k=0}^{\infty} S(k)x^{2k}$, equating like coefficients of $x^{2k}$ yields the desired result. ■

# A Bernoulli Identity

**Lemma 2**: For any $m \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{m} \binom{2m+1}{2k}(1 - 2^{2k-1})B_{2k} = 0.$

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Proof (sketch):
Substitute the generating function $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ and the power series for $e^t$ into $\frac{t}{e^t-1} = \frac{1+e^t}{2} \cdot \frac{2t}{e^{2t}-1}$.

This can be rewritten as
$\sum_{r=1}^{\infty} B_r(1 - 2^{r-1})\frac{t^r}{r!} = \sum_{r=0}^{\infty} \sum_{j=0}^{r} \binom{r}{j} 2^{j-1} B_j \frac{t^r}{r!}$.

By equating the coefficients of $t^{2m+1}$, using $\sum_{j=0}^{r} \binom{r}{j} B_j = B_r$, and noting that $B_{2k+1} = 0$, the result follows. ■

# A proof for $\zeta(2m)$ using telescoping sums

- Define for non-negative integers $k$ and $n$:

$$I_{k,n} = \int_0^{\frac{\pi}{2}} x^{2k} \cos^{2n} x \, dx.$$

- Observe that $A_n = I_{0,n}$, $B_n = I_{1,n}$, $C_n = I_{2,n}$.
- Initial values: $I_{k,0} = \dfrac{\pi^{2k+1}}{2^{2k+1}(2k+1)}$
- Recurrences: $I_{0,n} = \dfrac{2n-1}{2n} I_{0,n-1}$.

$$I_{k,n} = \frac{1}{(2k+2)(2k+1)}\left(2n(2n-1)I_{k+1,n-1} - 4n^2 I_{k+1,n}\right).$$

# A proof for $\zeta(2m)$ using telescoping sums, continued

- Divide both sides of the last recurrence by $n^2 I_{0,n}$ and substitute the first recurrence:

$$\frac{I_{k,n}}{n^2 I_{0,n}} = \frac{4}{(2k+2)(2k+1)}\left(\frac{I_{k+1,n-1}}{I_{0,n-1}} - \frac{I_{k+1,n}}{I_{0,n}}\right).$$

- Note that we have a family of recurrences as $k$ varies.

# A proof for $\zeta(2m)$ using telescoping sums, continued

- Let $S_N(0) = 1$, and for $k > 0$,
  $S_N(k) = \displaystyle\sum_{1 \le i_1 \le \cdots \le i_k \le N} \frac{1}{i_1^2 \cdot \ldots \cdot i_k^2}$, a truncated sum of $S(k)$.

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  $S_N(k) = \sum_{1 \le i_1 \le \cdots \le i_k \le N} \frac{1}{i_1^2 \cdot \ldots \cdot i_k^2}$, a truncated sum of $S(k)$.
- Telescoping step - Summing over $n$ and back substituting these equations (as $k$ varies) yields

$$\sum_{k=0}^{m} \frac{(-1)^k (2m)!}{2^{2k}(2m-2k)!} \cdot \frac{I_{m-k,0}}{I_{0,0}} \cdot S_N(k) = \frac{I_{m,N}}{I_{0,N}}.$$

# A proof for $\zeta(2m)$ using telescoping sums, continued

- Let $N \to \infty$, noting that $\frac{I_{m,N}}{I_{0,N}} \to 0$:

$$\sum_{k=0}^{m} \frac{(-1)^k (2m)!}{2^{2k}(2m-2k)!} \cdot \frac{I_{m-k,0}}{I_{0,0}} \cdot S(k) = 0.$$

- Apply Lemma 1 and solve for $\zeta(2m)$ (taking $\zeta(0) = -\frac{1}{2}$ as shorthand):

$$\zeta(2m) = \frac{1}{1-2^{1-2m}} \cdot \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1}\pi^{2m-2k}}{(2m-2k+1)!} \cdot (1-2^{1-2k})\zeta(2k).$$

- The desired result now follows from Lemma 2 and strong induction. ■

# Recommended reading


- *Euler's Solution of the Basel Problem - The Longer Story* http://www.math.uiuc.edu/ reznick/sandifer.pdf
- R. Chapman, *Evaluating* $\zeta(2)$: http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf
- D. Daners, A short elementary proof of $\sum 1/k^2 = \pi^2/6$, *Mathematics Magazine* 85(2012) 361-364
- T. Osler, Finding $\zeta(2p)$ from a Product of Sines, *The American Mathematical Monthly* 111(2004) 52-54
- F. Beukers, E. Calabi, and J. Kolk, Sums of generalized harmonic series and volumes, *Nieuw Archief Wiskunde* (4) 11 (1993) 217-224.

**Thank you!**