The Second Derivative Test in $n$ variables.

Brian D. Sittinger
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1 Introduction.

Stewart’s *Calculus* textbook does a fine job of addressing the differential calculus of functions of $n$ variables $f : \mathbb{R}^n \to \mathbb{R}$ in the cases of $n = 2$ or $3$. (As such, it’s usually easy to guess how these formulas generalise for arbitrary $n$.) However, it is curious that Stewart only covers the second derivative test in the case of $n = 2$ variables. After reviewing the cases of $n = 1$ and $2$, I’ll address what happens in general.

2 Second Derivative Test in 1 or 2 variables

Before stating the second derivative test as mentioned in Stewart, recall that for a function $y = f(x)$, the second derivative test uses concavity of the function at a critical point to determine whether we have a local maximum or minimum value at the said point.

**Second Derivative Test, Single variable case:**

Suppose that $f''$ is continuous near $c$, where $f'(c) = 0$ (that is, $c$ is a critical point of $f$).

(a) If $f''(c) > 0$, then $f$ has a local minimum at $c$.

(b) If $f''(c) < 0$, then $f$ has a local maximum at $c$. ■

Note that the theorem is silent in the case that $f''(c) = 0$. In such a case, one has to try a different method to determine the nature of the critical point.
Here is the analogous statement for two variables.

**Second Derivative Test, Two variable case:**
Suppose that the second partial derivatives of \( f : \mathbb{R}^2 \to \mathbb{R} \) are continuous on a disk with centre \((a, b)\), where \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \) (that is, \((a, b)\) is a critical point of \( f \)).

Let \( H \) denote the “Hessian” matrix of second partial derivatives
\[
H = \begin{bmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{bmatrix},
\]
and let \( D_1 = f_{xx} \) and \( D_2 = \det H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \).

(a) If \( D_1(a, b) > 0 \) and \( D_2(a, b) > 0 \), then \( f \) has a local minimum at \((a, b)\).
(b) If \( D_1(a, b) < 0 \) and \( D_2(a, b) > 0 \), then \( f \) has a local maximum at \((a, b)\).
(c) If \( D_2(a, b) < 0 \), then \( f \) has a saddle point at \((a, b)\).

A couple of remarks are in order. First, I write \( D_2 \) in terms of a determinant because not only is it easier to remember (which is always a good thing), but also it helps to point the way to further generalisations. Second, the test is inconclusive when \( D_2(a, b) = 0 \).

**Example:** Find and classify the critical points of the function \( f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1 \).

**Solution:** First, we set the first partial derivatives equal to zero:
\[
f_x = 6xy - 6x = 0 \quad \text{and} \quad f_y = 3x^2 + 3y^2 - 6y = 0.
\]
This yields four critical points \((0, 0), (0, 2), (1, 1), (-1, 1)\). Next, we compute the matrix of second partial derivatives:
\[
H = \begin{bmatrix}
  6y - 6 & 6x \\
  6x & 6y - 6
\end{bmatrix}.
\]
Now, we apply the second derivative test to each critical point.

1. At \((0, 0)\), \( D_1(0, 0) = -6 < 0 \), \( D_2(0, 0) = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0 \). Hence, \( f \) has a local maximum at \((0, 0)\).
2. At \((0, 2)\), \( D_1(0, 2) = 6 > 0 \), \( D_2(0, 0) = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0 \). Hence, \( f \) has a local minimum at \((0, 0)\).
3. At \((\pm 1, 1)\), \( D_2(\pm 1, 1) = \begin{vmatrix} 0 & \pm 6 \\ \pm 6 & 0 \end{vmatrix} = -36 < 0 \). Hence, \( f \) has saddle points at \((\pm 1, 1)\).
3 Second Derivative Test in 3 or more variables

By using the Hessian matrix, stating the second derivative test in more than 2 variables is not too difficult to do. Before stating the general theorem, we will first state it in 3 variables (so the pattern is clear) and work an example.

**Second Derivative Test, Three variable case:**
Suppose that the second partial derivatives of \( f : \mathbb{R}^3 \to \mathbb{R} \) are continuous on a ball with centre \((a, b, c)\), where \( f_x(a, b, c) = 0 \) and \( f_y(a, b, c) = 0 \), and \( f_z(a, b, c) = 0 \) (that is, \((a, b, c)\) is a critical point of \( f \)).

Let \( H \) denote the Hessian matrix of second partial derivatives
\[
H = \begin{bmatrix}
  f_{xx} & f_{xy} & f_{xz} \\
  f_{yx} & f_{yy} & f_{yz} \\
  f_{zx} & f_{zy} & f_{zz}
\end{bmatrix},
\]
and let \( D_1 = f_{xx} \), \( D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \), and \( D_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} \).

(a) If \( D_1(a, b, c) > 0 \), \( D_2(a, b, c) > 0 \), and \( D_3(a, b, c) > 0 \), then \( f \) has a local minimum at \((a, b, c)\).
(b) If \( D_1(a, b, c) < 0 \), \( D_2(a, b, c) > 0 \), and \( D_3(a, b, c) < 0 \), then \( f \) has a local maximum at \((a, b, c)\).
(c) In any other case where \( D_3(a, b, c) \neq 0 \), \( f \) has a saddle point at \((a, b, c)\).

**Example:** Find and classify the critical points of the function \( f(x, y, z) = x^2 + y^2 + 7z^2 - xy - 3yz \).

**Solution:** First, we set the first partial derivatives equal to zero:
\[
f_x = 2x - y = 0, \quad f_y = 2y - x - 3z = 0, \quad \text{and} \quad f_z = 14z - 3y = 0.
\]
This yields exactly one critical point \((0, 0, 0)\). Next, we compute the matrix of second partial derivatives:
\[
H = \begin{bmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -3 \\
  0 & -3 & 14
\end{bmatrix}.
\]
Now, we apply the second derivative test to the critical point \((0, 0, 0)\).
\[
D_1(0, 0, 0) = 2 > 0, \quad D_2(0, 0, 0) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0, \quad \text{and} \quad D_3(0, 0, 0) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{vmatrix} = 24 > 0.
\]
Hence, \( f \) has a local minimum at \((0, 0, 0)\).
Now, we can state the general version of the second derivative test. Observe how this theorem indeed does summarise the forms of this test given in the specific cases. For brevity, let $\vec{x} = (x_1, x_2, ..., x_n)$, and use the gradient of $f$ to concisely list the partial derivatives.

**Second Derivative Test, the general $n$ variable version:**
Suppose that the second partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ are continuous on a ball with centre $\vec{c}$, where $\nabla f(\vec{c}) = 0$ (that is, $\vec{c}$ is a critical point of $f$).

Let $H$ denote the Hessian matrix of second partial derivatives, and for each $k = 1, 2, ..., n$, let $D_k$ denote the determinant of the Hessian in the variables $x_1, x_2, ..., x_k$. Assume that $|H(\vec{c})| \neq 0$.

(a) If $D_k(\vec{c}) > 0$ for all $k = 1, 2, ..., n$, then $f$ has a local minimum at $\vec{c}$.
(b) $(-1)^k \cdot D_k(\vec{c}) > 0$ for all $k = 1, 2, ..., n$, then $f$ has a local maximum at $\vec{c}$.
(c) Otherwise, $f$ has a saddle point at $\vec{c}$. ■

This theorem is usually proved using the quadratic approximation of the (multivariable) Taylor Series for $f$ centred at $\vec{x} = \vec{c}$ and understandably involves a good amount of Linear Algebra. The curious student may consult an Advanced Calculus textbook for a proof of this theorem.