Math 351: The First Homework Set.

1. (a) If \( p \) is a prime number, then prove that \( \sqrt{p} \) is irrational.
   You can assume the fact that if \( ab \) is divisible by \( p \) for some integers \( a \) and \( b \),
   then \( a \) or \( b \) is divisible by \( p \).
   
   (b) Prove that \( \sqrt{6} \) is irrational.

2. If \( r \) is a nonzero rational number and \( x \) is an irrational number, prove that \( r + x \) and \( rx \) are both irrational. (You may assume that \( \mathbb{Q} \) is closed under addition and multiplication.)

3. Show that \( \sqrt{2} + \sqrt{3} \) is irrational.

4. Show that \( \log_2 3 \) is irrational.

5. Suppose that \( x \) is a nonnegative real number such that \( x < \epsilon \) for any choice of \( \epsilon > 0 \).
   Then, prove that \( x = 0 \). (Try proving this by contradiction.)

6. Let \( A \) be a nonempty subset of \( \mathbb{R} \). Suppose that \( \alpha \) is a lower bound of \( A \) and \( \beta \) is an upper bound of \( A \). Show that \( \alpha \leq \beta \).

7. Let \( A \) be a nonempty subset of \( \mathbb{R} \).
   
   (a) Write a formal definition for the infimum (greatest lower bound) of \( A \).
   
   (b) Prove the following analogue of Lemma 1.3.7 for the infimum:
       
       Assume \( s \in \mathbb{R} \) is a lower bound for \( A \). Then, \( s = \inf A \) if and only if, for every choice of \( \epsilon > 0 \), there exists an element \( a \in A \) satisfying \( s + \epsilon > a \).