Math 351: Review for the Final Exam.

Be able to state and work with:

1. Compact set (in $\mathbb{R}$)
2. Continuity (pointwise and uniform)
3. Derivative, Mean Value Theorem
4. Sequence/Series of functions: Pointwise and uniform convergence and related results
5. Riemann Integral and conditions for integrability
6. Know the other definitions, too...

Be able to prove:

1. The first two parts of the Algebraic Limit Theorem
2. Heine-Borel Theorem
3. Theorem 4.4.2 (Continuous image of a compact set)
4. Intermediate Value Theorem
5. Theorem 7.2.9 (Continuous functions and Riemann Integrability)
6. Theorem 6.2.6 (Uniform convergent sequence of continuous functions)

Extra practice problems:

1. Show that $\sqrt{2}$ is irrational.
2. Show that $\lim_{x \to 0} 5x^4 \cos\left(\frac{\pi}{x}\right) = 0$.
3. Use the $\delta - \varepsilon$ definition to show that $f(x) = x^2 - 4x + \sqrt{x}$ is continuous at $x = 4$.
4. Let $f_n(x) = \frac{nx}{1+n^2x^2}$. Find the pointwise limit of $f_n$ for all $x \in [0, \infty)$. Is the convergence uniform on $[0, \infty)$? Explain.
5. Show that $f(x) = \sum_{n=0}^{\infty} \frac{\sin(n^7e^{-7x^3})}{7^n + 2009}$ is continuous for all $x \geq 0$.
6. Give an example of a function defined on $\mathbb{R}$ which is only continuous at 2 points. Can you generalise this to finitely many points? Countably many points?
7. Show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. 
Proofs of Theorems to be internalised for the Final Exam.

1. Intermediate Value Theorem: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $y$ is between $f(a)$ and $f(b)$. Then, $f(c) = y$ for some $c$ between $a$ and $b$.

Proof (via Bisection):
Again without loss of generality, assume that $f(a) < f(b)$. Let $I_1 = [a, b]$ and let $x_1$ be the midpoint of $I_1$. If $f(x_1) = y$, we are done. If $f(x_1) < y$, then let $I_2 = [x_1, b]$. Otherwise, let $I_2 = [a, x_1]$ and proceed inductively.
If this process never stops, let $\{a_k\}$ and $\{b_k\}$ be the sequences of left and right endpoints of $I_k$. Then, we have the following:

(a) $\{a_k\}$ is increasing and $\{b_k\}$ is decreasing,
(b) $f(a_k) < y < f(b_k)$ for all $k \in \mathbb{N}$,
(c) $I_1 \supset I_2 \supset I_3 \supset \ldots$, and
(d) $b_k - a_k = 2^{-k}(b - a)$.

By (a) and (c), $\{a_k\}$ is monotonically increasing and bounded above. Then the Monotone Convergence Theorem implies that $\{a_k\} \to s$ for some $s \in \mathbb{R}$. Since $f$ is continuous, we have $\{f(a_k)\} \to y$. Since $f(a_k) < y$ by (b), we conclude that $f(s) \leq y$ by the Order Limit Theorem. Similarly, $\{b_k\}$ has a limit $t$ and $f(t) \geq y$.

Since $a_k \leq s$ and $b_k \geq t$ for all $k \in \mathbb{N}$, (d) yields $t - s \leq b_k - a_k = 2^{-k}(b - a)$ for all $k \in \mathbb{N}$. In addition, since $t \geq s$, we can rewrite this as $0 \leq t - s \leq 2^{-k}(b - a)$ for all $k \in \mathbb{N}$. Finally, since $\lim_{k \to \infty} 2^{-k}(b - a) = 0$, the Squeeze Theorem implies that $s = t$. (This common value is $c$ as asserted in the conclusion to the theorem). Therefore, $f(s) = y = f(t)$ as desired. ■