Math 451, Review for the Midterm.

Here are a few practice problems in the mould of those which can appear on the exam itself. Also look at related homework problems for extra practice as necessary. Make sure you know the statements of key theorems and definitions well!

1. Compute \((1 + i\sqrt{3})^{20}\), as well as find all values of \((-4i)^{1/4}\).

2. Compute the twelve twelfth roots of unity (simplify these to \(a + bi\) form). Do you notice anything familiar about any of these results?

3. Compute (i) \(\exp(-1 + \pi i/4)\), (ii) \(\log(8 + 15i)\).

4. Compute all values of \((7i)^{1/2}\). Which of these is the principal value?

5. Given \(f(z) = z^2 \Re z\), determine whether \(f\) is differentiable at \(z = 0\) by (a) The definition of the derivative, and (b) The Cauchy-Riemann Equations.

6. Find a harmonic conjugate \(v(x, y)\) of \(u(x, y) = x - y + 2x^2 - 2y^2\). Then, find a function \(f(z)\) such that \(f(z) = u(x, y) + iv(x, y)\).

7. Find the 6th derivative of \(f(z) = \exp((-1 + i\sqrt{3})z)\).

8. Using the definition of \(e^{i\theta}\), find expressions for \(\cos(5\theta)\) and \(\sin(5\theta)\). The Binomial Theorem may be useful.

9. Let \(f = u + iv\) be analytic on a domain \(D\), where \(u\) and \(v\) are real-valued. Suppose that there exist constants \(a, b, c \in \mathbb{R}\) such that \(a^2 + b^2 \neq 0\) and \(au + bv = c\) in \(D\). Show that \(f\) is constant in \(D\). (Hint: Use the Cauchy-Riemann Equations.)
Solutions:

1. \((1 + i\sqrt{3})^{20} = (2e^{\pi i/3})^{20} = 2^{20}e^{20\pi i/3} = 2^{19}(-1 + i\sqrt{3})\)

\((-4i)^{1/4} = (4e^{-\pi i/2+2\pi ik})^{1/4} = 4^{1/4}e^{-\pi i/8+\pi ik/2} \text{ for } k = 0, 1, 2, 3. \) (None of these simplify too nicely.)

2. \(1^{1/12} = \exp(2\pi ik/12) \text{ for } k = 0, 1, 2, ..., 11.\) Simplifying yields the twelve 12th roots of unity: \(\pm 1, \pm i, \pm (\frac{1+i\sqrt{3}}{2}), \pm (\frac{1-i\sqrt{3}}{2}), \pm (\frac{\sqrt{3}+i}{2}), \pm (\frac{\sqrt{3}-i}{2}).\)

These form the vertices of a regular dodecagon. Moreover, which of these are cube roots, fourth roots, or even sixth roots of unity?

3. (i) \(\exp(-1 + \pi i/4) = e^{-1}(\sqrt{2} + i\frac{\sqrt{2}}{2}),\)

(ii) \(\log(8 + 15i) = \ln|8 + 15i| + i\arg(8 + 15i) = \ln 17 + i(\arctan(\frac{15}{8}) + 2\pi k) \text{ for any } k \in \mathbb{Z}.

4. \((7i)^{1/2} = \exp[(\frac{i}{2}) \log(7i)] = \exp\{\frac{i}{2}[\ln 7 + (\frac{n}{2})(4k + 1)]\} = \exp\{\frac{i}{2}\ln 7 - (\frac{n}{4})(4k + 1)\} \text{ for any } k \in \mathbb{Z}.

Principal value occurs when \(k = 0: e^{-\pi/4}(\cos(\frac{1}{2}\ln 7) + i\sin(\frac{1}{2}\ln 7)).\)

5. \(f(z) = z^2 \text{ Re } z \text{ is differentiable at } z = 0:\)

(i) \(f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} (\Delta z)^2 \text{ Re } (\Delta z) = 0.\)

(ii) First of all, check that \(f(z) = u(x,y) + iv(x,y), \text{ where } u = x^2 - xy^2 \text{ and } v = 2x^2y. \) Then, verify that \(u_x = v_y \text{ and } u_y = -v_x \text{ at } (0,0).\)

6. \(v(x,y) = 4xy + y + x, \text{ and } f(z) = z + iz + 2z^2.\)

7. \(f^{(6)}(z) = (-1 + i\sqrt{3})^6 \exp((-1 + i\sqrt{3})z) = 64\exp((-1 + i\sqrt{3})z).\)

8. Using the definition of \(e^{i\theta}, \text{ find expressions for } \cos(5\theta) \text{ and } \sin(5\theta). \) The Binomial Theorem may be useful.

On one hand, \(e^{5i\theta} = (\cos \theta + i \sin \theta)^5 = \cos(5\theta) + i \sin(5\theta).\)

On the other hand, \((\cos \theta + i \sin \theta)^5 = \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta.\)

Now, equate real and imaginary parts, and you are done.

9. Differentiating with respect to \(x\) as well as \(y\) yields \(au_x + bu_x = 0 \text{ and } au_y + bv_y = 0.\)

Rewrite the second equation as \(-au_x + bu_x = 0 \text{ by the Cauchy-Riemann Equations.} \)

Solving for \(u_x\) and \(v_x\) shows that \(u_x = 0 \text{ and } v_x = 0 \text{ on } D. \) Hence, \(f'(z) = u_x + iv_x = 0 \text{ on } D, \) and thus \(f\) is constant on \(D.\)