Math 499, Problem Set #1.

1. Solve the following two systems as efficiently as possible.
   (a) $xy = 12\sqrt{6}, yz = 54\sqrt{2}, xz = 48\sqrt{3}$.
   (b) $(x + y)(x + y + z) = 66$, $(y + z)(x + y + z) = 99$, $(x + z)(x + y + z) = 77$.

2. (a) Suppose that $a, b, c$ are the roots of a cubic polynomial with leading coefficient 1. What is the equation of the cubic equation in terms of $a, b, c$?
   (b) Solve the system $x + y - z = 0, xz - xy + yz = 27, xyz = 54$.

3. Factor $x^7 + 1$ as the product of linear and quadratic factors with real coefficients.

4. If $A$ and $B$ are square matrices of the same dimension and $AB^2 - A$ is invertible, show that $BA - A$ is also invertible.

5. Let $a, b > 0$. Show that $\lim_{n \to \infty} (a^n + b^n)^{1/n} = \max\{a, b\}$.

6. Show that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = e^{-x}$ for all $x \in \mathbb{R}$. (It may be useful to first establish that if $f \circ f$ has a unique fixed point, then so does $f$.)

7. Show that $\int_1^\infty \left( \frac{1}{\sqrt{t^3} + 1} - \frac{1}{t + 1} \right) dt$ converges.