Math 499, Problem Set #2.

1. Find (and derive) a formula for the area of the of a regular \( n \)-sided polygon whose side has length \( s \). You may find it useful to decompose the polygon into triangles!

2. Without using a calculator, determine which is greater: \( \pi^e \) or \( e^\pi \).

3. Factor \( n^4 + n^2 + 1 \) as the product of two quadratic polynomials with real coefficients.
   Use this to evaluate \( \sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1} \) as a telescoping sum.

4. Show that \( \frac{1}{x} \) cannot be written as the derivative of a rational function (that is, of the form \( \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomials).

5. Show that if 5 points are all in, or on, a square whose sides have length 1, then some pair of them will be no further than \( \sqrt{2} \) apart.

6. Suppose that \( f : \mathbb{R} \to [0, \infty) \) satisfies \( [f(x + y)]^2 - [f(x - y)]^2 = 4f(x)f(y) \) for any real numbers \( x \) and \( y \). Prove by induction (or otherwise) that \( f(nx) = nf(x) \) for any integer \( n \). (It may help to find the relation between \( f(x) \) and \( f(-x) \).)

7. (a) For any nonnegative real numbers \( a \) and \( b \), show that \( \frac{a+b}{2} \geq \sqrt{ab} \).
   (Its generalization is the useful Arithmetic Mean-Geometric Mean Inequality: For any \( a_1, ..., a_n \geq 0 \), then \( \frac{a_1 + ... + a_n}{n} \geq \sqrt[1/a_1 \cdots a_n]{a_1} \cdots a_n \). You need not prove this here.)
   (b) Show that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} \) also converges.