Math 499, Problem Set #9.

1. Prove that for any $r \in \mathbb{Z}_{\geq 0}$, \( \sum_{j=1}^{n} j(j+1)(j+r) = \frac{n(n+1)(n+r+1)}{r+2} \).

2. Show that $x^3 + y^5 = z^7$ has infinitely many positive integer solutions.

3. Suppose that $f : [0, 1] \to [0, 1]$ is continuous. Show that $f(x) = x$ for some $x \in [0, 1]$. (Hint: Consider applying the Intermediate Value Theorem to $g(x) = f(x) - x$.)

4. Evaluate the following limits:
   \begin{align*}
   (a) \quad & \lim_{x \to \infty} (\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 5x^2}) \\
   (b) \quad & \lim_{x \to \infty} [2\sqrt{x} (\sqrt{x + 1} - \sqrt{x})]^x.
   \end{align*}

5. Show that \( \lim_{x \to 0} \frac{1 - \cos x \cdot \cos(2x) \cdot \ldots \cdot \cos(nx)}{x^2} = \frac{n(n+1)(2n+1)}{12} \) for any $n \in \mathbb{N}$.

6. Find $\frac{dy}{dx}$ if $y = x^{x^{x^{\ldots}}}$ (Note that this is an infinite tower of exponents.)

7. Show that any infinite set of nonintersecting discs drawn in the plane ($\mathbb{R}^2$) is countable.

8. Let $S$ denote the set of positive integers which do not have 7 as a digit. Show that \( \sum_{n \in S} \frac{1}{n} \) converges.

9. Find the region $\mathcal{R}$ which maximizes $\iiint_{\mathcal{R}} (1 - 4x^2 - 9y^2 - z^2) \, dV$. Then, evaluate the integral on this region. (Using the change of coordinates $x = \frac{1}{2} \rho \cos \theta \sin \phi$, $y = \frac{1}{3} \rho \sin \theta \sin \phi$, and $z = \rho \cos \phi$ may be useful.)

10. Let $f(x) = x \sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$. Show that the region between $f$ and the $x$-axis on $[0, 1]$ has finite area but infinite arc length.

11. Show that \( \int_{0}^{1} (\sqrt{1 - x^2} - \sqrt{1 - x^3}) \, dx = 0 \). (Thinking of the integral(s) as areas may be useful.)

12. Evaluate the Gaussian Integral $G = \int_{-\infty}^{\infty} e^{-x^2} \, dx$ as follows:
   
   On one hand, the volume between the surface $z = e^{-(x^2+y^2)}$ and the $xy$-plane is $G^2$. On the other hand, this volume can be thought of as a volume of revolution by revolving $y = e^{-x^2}$ about the $y$-axis; evaluate this by Calculus II methods.