Math 499, Problem Set #9 Solutions.

1. Prove that for any \( r \in \mathbb{Z}_{\geq 0}, \sum_{j=1}^{n} j(j+1)...(j+r) = \frac{n(n+1)...(n+r+1)}{r+2} \).

Solution: A proof by induction on \( n \) is straightforward. For the sake of variety, I will prove this by rewriting this as a telescoping sum. Observe that \( j(j+1)...(j+r+1) - (j-1)j...(j+r) = j(j+1)...(j+r)[(j+r+1) - (j-1)] = (r+2)j(j+1)...(j+r) \).

So, \( \sum_{j=1}^{n} j(j+1)...(j+r) = \frac{1}{r+2} \cdot \sum_{j=1}^{n} [j(j+1)...(j+r+1) - (j-1)j...(j+r)] = \frac{n(n+1)...(n+r+1)}{r+2} \), by telescoping. \( \blacksquare \)

2. Show that \( x^3 + y^5 = z^7 \) has infinitely many positive integer solutions.

Solution: There are many ways to solve this (ask me for more details if you are curious). I will display a one-parameter family of solutions. Due the the exponents, let \( x = 2^{5k} \) and \( y = 2^{3k} \) for some non-negative integer \( k \) to be determined. Substituting this into the equation, we obtain \( 2^{15k} + 2^{15k} = 2^{15k+1} = z^7 \). Assuming that \( z \) is a power of 2, we need \( 15k+1 \equiv 0 \pmod{7} \), and thus \( k \equiv -1 \pmod{7} \).

Writing \( k = 7n - 1 \) for any \( n \in \mathbb{N} \), we have the infinite family of positive integer solutions \( (x, y, z) = (2^{5n-5}, 2^{21n-3}, 2^{105n-14}/7) = (2^{35n-5}, 2^{21n-3}, 2^{15n-2}). \) \( \blacksquare \)

3. Suppose that \( f : [0,1] \to [0,1] \) is continuous. Show that \( f(x) = x \) for some \( x \in [0,1] \).

Solution: If \( f(0) = 0 \) or \( f(1) = 1 \), then we are done. So, we will hereafter assume otherwise. Now, consider \( g(x) = f(x) - x \), which is also continuous on \([0,1]\) (since \( f(x) \) and the identity function are). Since \( g(0) = f(0) - 0 > 0 - 0 = 0 \) and \( g(1) = f(1) - 1 < 1 - 1 = 0 \), we conclude by the Intermediate Value Theorem that \( g(x) = 0 \) for some \( x \in (0,1) \). That is, \( f(x) = x \) for some \( x \in (0,1). \) \( \blacksquare \)

4. Evaluate the following limits:

(a) \( \lim_{x \to \infty} (\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 5x^2}) \)

Solution: By factoring \( x \) from both radicals we have
\( \lim_{x \to \infty} (\sqrt{x^2 + 1} - \sqrt[3]{x^3 + 5x^2}) = \lim_{x \to \infty} \left[ x \sqrt{1 + \frac{1}{x^2}} - x \sqrt[3]{1 + \frac{5}{x}} \right] \).

Now, rewrite this as \( \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}} - \sqrt[3]{1 + \frac{5}{x}}}{\frac{1}{x}} \).

Letting \( t = \frac{1}{x} \), this limit transforms to \( \lim_{t \to 0^+} \frac{\sqrt{1 + t^2} - \sqrt[3]{1 + 5t}}{t} \).
Now, L'Hôpital's Rule yields \( \lim_{t \to 0^+} \frac{t(1 + t^2)^{-1/2} - \frac{5}{3}(1 + 5t)^{-2/3}}{1} = -\frac{5}{3}. \)

(b) \( \lim_{x \to \infty} [2\sqrt{x} \left( \sqrt{x + 1} - \sqrt{x} \right)]^x. \)

**Solution:** Using conjugates, we can rewrite the limit (call it \( L \)) as \( \lim_{x \to \infty} \frac{2\sqrt{x}}{\sqrt{x + 1} + \sqrt{x}}. \) Now, take natural logarithms:

\[
\ln L = \lim_{x \to \infty} x \cdot \ln \left( \frac{2\sqrt{x}}{\sqrt{x + 1} + \sqrt{x}} \right) = \lim_{x \to \infty} \frac{\ln \left( \frac{2}{\sqrt{1 + \frac{1}{x}}} \right)}{\frac{1}{x}}.
\]

Now, let \( t = \frac{1}{x} \):

\[
\ln L = \lim_{t \to 0^+} \frac{\ln \left( \frac{2}{\sqrt{1 + t}} \right)}{\frac{1}{t}} = \lim_{t \to 0^+} \frac{2 - \ln(1 + \sqrt{1 + t})}{t} = -\frac{1}{4}.
\]

Finally, \( L = e^{\frac{1}{4}}. \)

5. Show that \( \lim_{x \to 0} \frac{1 - \cos x \cdot \cos(2x) \cdot \ldots \cdot \cos(nx)}{x^2} = \frac{n(n + 1)(2n + 1)}{12} \) for any \( n \in \mathbb{N} \).

**Solution:** Using the familiar Maclaurin Series for cosine, note that

\[
\cos x \cdot \cos(2x) \cdot \ldots \cdot \cos(nx) = 1 - \frac{1}{2^n} \cdot (1^2 + 2^2 + \ldots + n^2)x^2 + O(x^4).
\]

Since \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \), we see that

\[
1 - \cos x \cdot \cos(2x) \cdot \ldots \cdot \cos(nx) = \frac{n(n + 1)(2n + 1)}{12}x^2 + O(x^4).
\]

So, the limit reduces to \( \lim_{x \to 0} \left[ \frac{n(n + 1)(2n + 1)}{12} + O(x^2) \right] = \frac{n(n + 1)(2n + 1)}{12}. \)

6. Find \( \frac{dy}{dx} \) if \( y = x^{x^x} \). (Note that this is an infinite tower of exponents.)

**Solution:** Note that the given equation is equivalent to \( y = x^y \). Take natural logarithms of both sides: \( \ln y = y \ln x \). Differentiate both sides with respect to \( x \) and solve for \( \frac{dy}{dx} \): We obtain \( \frac{1}{y} \frac{dy}{dx} = \ln x \frac{dy}{dx} + \frac{y}{x} \). Hence, \( \frac{dy}{dx} = \frac{y^2}{x - xy \ln x}. \)

7. Show that any infinite set \( S \) of nonintersecting discs drawn in the plane \( (\mathbb{R}^2) \) is countable.

**Solution:** By density of \( \mathbb{Q} \) in \( \mathbb{R} \), each disk has a rational point in it; identify each disc with such a rational point. This yields a bijection between \( S \) and an infinite subset of the countable set \( \mathbb{Q}^2 \). Hence, the set in discussion is also countable.
8. Let $S$ denote the set of positive integers which do not have 7 as a digit.
Show that $\sum_{n \in S} \frac{1}{n}$ converges.

**Solution:** The idea is to group the terms by the number of digits they have. For $k$ digit numbers, there are $8 \cdot 9^{k-1}$ of them without 7’s (the 8 counts the leading digit (no 0 or 7), and the 9’s come from the remaining digits (no 7)). The terms in each grouping of the series is bounded above by $\frac{1}{10^{k-1}}$.

Hence, $\sum_{n \in S} \frac{1}{n}$ is bounded above by the convergent geometric series $\sum_{k=1}^{\infty} \frac{8 \cdot 9^{k-1}}{10^{k-1}}$, and thus converges by the Comparison Test. ■

9. Find the region $\mathcal{R}$ which maximizes $\iiint_{\mathcal{R}} (1 - 4x^2 - 9y^2 - z^2) \, dV$. Then, evaluate the integral on this region.

**Solution:** Use the ellipsoidal change of coordinates $x = \frac{1}{2} \rho \cos \theta \sin \phi$, $y = \frac{1}{3} \rho \sin \theta \sin \phi$, and $z = \rho \cos \phi$. Since this has jacobian $\frac{1}{6} \cdot \rho^2 \sin \phi$, the integral transforms to $\iiint_{\mathcal{R}'} (1 - \rho^2) \cdot \left(\frac{1}{6} \cdot \rho^2 \sin \phi \, dV'\right)$. To maximize this integral, we need to integrate it on $\rho \in [0, 1]$ (the maximal domain for which the integrand is never negative), and the maximal domains for $\theta$ and $\phi$; namely $[0, 2\pi]$, and $[0, \pi]$, respectively. Note that in terms of the original variables, this is the interior of the ellipsoid $4x^2 + 9y^2 + z^2 = 1$.

On this region, the integral attains maximal value

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 (1 - \rho^2) \cdot \left(\frac{1}{6} \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\right) = \frac{4\pi}{45}.$$ ■

10. Let $f(x) = x \sin\left(\frac{x}{2}\right)$ if $x \neq 0$ and $f(0) = 0$. Show that the region between $f$ and the $x$-axis on $[0, 1]$ has finite area but infinite arc length.

**Solution:** The area is finite by the Comparison Test for integrals, because $0 \leq |f(x)| \leq x$ on $[0, 1]$, and $\int_0^1 x \, dx = \frac{1}{2}$. As for the arc length, note that $f(x)$ has local extrema when $x = \frac{2}{(2n-1)\pi}$ for any $n \in \mathbb{N}$. Then, the arc length $L$ of the curve is bounded below by the sums of the lengths of the line segments connecting the local extrema:

$$L > \sum_{n=1}^{\infty} \sqrt{\left(\frac{2}{(2n-1)\pi} - \frac{2}{(2n+1)\pi}\right)^2 + \left(\frac{2}{(2n-1)\pi} + \frac{2}{(2n+1)\pi}\right)^2}.$$ 

This sum is bounded below by $\frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \sqrt{0 + \left(\frac{1}{2n-1} + 0\right)^2} = \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{2n-1}$, which is readily seen to diverge by the Integral Test (or Limit Comparison with the harmonic series). ■
11. Show that \( \int \limits_0^1 (\sqrt[3]{1-x^7} - \sqrt[3]{1-x^3}) \, dx = 0 \). (Thinking of the integral(s) as areas may be useful.)

**Solution:** Note that the area between \( y = \sqrt[3]{1-x^7} \) and the \( x \)-axis for \( x \in [0, 1] \) is equal to the area between \( x = \sqrt[3]{1-y^3} \) and the \( y \)-axis for \( y \in [0, 1] \), since they both represent the same region. A change in dummy variable now yields the result.

12. Evaluate the Gaussian Integral \( G = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) as follows:

On one hand, the volume between the surface \( z = e^{-(x^2+y^2)} \) and the \( xy \)-plane is \( G^2 \).

On the other hand, this volume can be thought of as a volume of revolution by revolving \( y = e^{-x^2} \) about the \( y \)-axis; evaluate this by Calculus II methods.

**Solution:** Following the hints, the volume between the surface \( z = e^{-(x^2+y^2)} \) and the \( xy \)-plane equals \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dy \, dx = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy = G^2 \).

Next, interpreting the region as that from revolving the region between \( y = e^{-x^2} \) and the positive \( x \)-axis, about the \( y \)-axis, the volume equals (by the Cylindrical Shell method) \( \int_{0}^{\infty} 2\pi xe^{-x^2} \, dx = -\pi e^{-x^2} \bigg|_{0}^{\infty} = \pi \). So \( G^2 = \pi \), and thus \( G = \sqrt{\pi} \), since the integrand of \( G \) is positive.