Math 582, The Fifth Homework Set.

For next class, skim through Chapter 5 in the text by Stewart and Tall.

1. Which of the following elements of \( \mathbb{Z}[i] \) are irreducible? (a) \( 1 + i \), (b) \( 3 - 7i \), (c) 5, (d) \( 12i \), (e) \( -4 + 5i \).

2. Is \( 10 = 2 \cdot 5 = (3 + i)(3 - i) \) an example of non-unique prime factorisation in \( \mathbb{Z}[i] \)? Explain your reasoning.

3. (a) Find all ideals in \( \mathbb{Z} \) which contain \( \langle 60 \rangle \). By direct examination of the possibilities, show that every ascending chain of ideals of \( \langle 60 \rangle \) terminates. (This is best done by just listing the longest chains without repeated ideals. There are 12 of these.)

(b) **Bonus:** Let \( n > 1 \) be an integer. How many ascending chain of ideals are there for \( \langle n \rangle \)? The answer will have something to do with the prime factorisation of \( n \).

4. Show that \( \mathbb{Z}[x_1, x_2, x_3, ...] \), a polynomial ring in infinitely many variables, is not noetherian.

5. Let \( D \) be any integral domain. Suppose that \( x \in D \) has a factorisation

\[
x = up_1...p_n
\]

where \( u \in D^* \) and \( p_1, ..., p_n \in D \) are primes. Show that given any factorisation

\[
x = vq_1...q_m
\]

where \( v \in D^* \) and \( q_1, ..., q_m \in D \) are irreducibles, then \( m = n \) and there exists a permutation \( \pi \) of \( \{1, 2, ..., n\} \) such that \( p_i \) and \( q_{\pi(i)} \) (for \( i = 1, ..., n \)) are associates.

6. Let \( p \) be a prime, and define \( \mathbb{Q}(p) \) to be the set of all rational numbers in reduced terms whose denominators are not divisible by \( p \). Explain why \( \mathbb{Q}(p) \) is an integral domain. Then, show that the only irreducible elements in \( \mathbb{Q}(p) \) are \( p \) and its associates.

7. Read the derivation of the integer solutions of \( y^2 + 2 = x^3 \) on p. 78 in the text. Explain this proof in your words, adding details as necessary. You may use the fact that \( \mathbb{Z}[\sqrt{-2}] \) has unique factorisation into prime numbers (as discussed in class).