GRE Math Subject Test #1 Solutions.

1. $\boxed{\text{B}}$ (Geometry) Note that all three points have $y = 0$. So, they lie on the xz-plane.

2. $\boxed{\text{C}}$ (Algebra) Straightforward; beware of negative numbers!

3. $\boxed{\text{B}}$ (Calculus) By Fubini's Theorem, $\int_0^1 \int_0^x xy\, dy\, dx = \int_0^1 \frac{x^3}{2} dx = \frac{1}{8}$.

4. $\boxed{\text{A}}$ (Calculus) By the Product Rule, this equals $ex^{e-1} \cdot e^x + x^e \cdot e^x = x^{e-1}e^{x+1} + x^e e^x$.

5. $\boxed{\text{A}}$ (Calculus) Integrating the first equation with respect to x and the second equation with respect to y yields $f(x, y) = x^2 + xy + g(y)$ and $f(x, y) = xy + y^2 + h(x)$ for some g, h. Comparing these yields $f(x, y) = x^2 + xy + y^2 + C$ for some constant C.

6. $\boxed{\text{C}}$ (Calculus) Remember that the derivative is the slope of the tangent line at a given point. In this case, since f consists of straight lines, the graph of its derivative consists of horizontal line segments whose y-coordinates are the slopes from the lines comprising f.

7. $\boxed{\text{A}}$ (Calculus) The upper line segment is that of $y = x + 2$, and the bottom "v" graph is that of $y = |x|$. Hence, we want the integral representing the area between $y = x + 2$ and $y = |x|$ for $x \in [-1, 1]$.

8. $\boxed{\text{E}}$ (Calculus) This series diverges by the n-th Term Test, because $\lim_{n\to\infty} \frac{n}{n+1} = 1$, which is nonzero. Since the terms are all positive, it diverges to ∞.

9. $\boxed{\text{E}}$ (Probability) For each digit, there are 10 possibilities, 9 of which are nonzero. So, the multiplication principle shows that the probability equals $(\frac{9}{10})^k$.

10. $\boxed{\text{C}}$ (Linear Algebra) Simply compute M via $M = (MC)C^{-1}$. This yields $\begin{bmatrix} 18 & 15 \\ 13 & 5 \end{bmatrix}$, which spells *ROME*.

11. $\boxed{\text{B}}$ (Trigonometry) Use the cofunction identity $\arcsin x + \arccos x = \frac{\pi}{2}$.

12. $\boxed{\text{B}}$ (Calculus) $\int_0^\pi e^{\sin^2 x} e^{\cos^2 x}\, dx = \int_0^\pi e^1\, dx = e\pi$.

13. $\boxed{\text{D}}$ (Calculus) Note that the limit is infinite as $x \to 2$ from either side.

14. $\boxed{\text{A}}$ (Algebra) No comments!

15. $\boxed{\text{E}}$ (Linear Algebra) Since f is a linear transformation, we have $f(3, 5) = f(5(1, 1) + 2(-1, 0)) = 5f(1, 1) + 2f(-1, 0) = 5 \cdot 1 + 2 \cdot 2 = 9$.

16. $\boxed{\text{B}}$ (Calculus) This is the geometric interpretation of the Mean Value Theorem.

17. $\boxed{\text{C}}$ (Abstract Algebra) Just use the definition of $*$. Clearly, $a * b = b * a$ for any $a, b \in \mathbb{Q}$; so (I) is true. Next, note that $a * 0 = 0 * a = a$ for all $a \in \mathbb{Q}$, so (II) is true (0 is the identity element). Next, if we try to find a^{-1} so that $a * a^{-1} = 0$ for all $a \in \mathbb{Q}$, we find that $a^{-1} = \frac{-a}{1+2a}$, which is not defined for $a = \frac{-1}{2}$. So, (III) is false.

18. $\boxed{\textbf{D}}$ (Abstract Algebra) Since $(ab)^2 = (ab)(ab) = abab$, left cancelation by a and right cancelation by b on $(ab)^2 = a^2b^2$ yields $ba = ab$ for all $a, b \in G$.

19. $\boxed{\textbf{D}}$ (Calculus) Note that $f'(x) = e^x - c$ set equal to 0 yields the critical point at $x = \ln c$. Since $f''(\ln c) > 0$, we have a local minimum by the Second Derivative Test.

20. $\boxed{\textbf{C}}$ (Algebra) The only periodic polynomials are constants.

21. $\boxed{\textbf{A}}$ (Algebra) Replace x with e^x to obtain $f(x) = \sqrt{e^x} = e^{x/2}$.

22. $\boxed{\textbf{B}}$ (Calculus) By Fubini's Theorem, $\int_0^1 \int_0^{\sin y} \frac{1}{\sqrt{1-x^2}}\, dx\, dy = \int_0^1 y\, dy = \frac{1}{2}$. The key fact here is that $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$.

23. $\boxed{\textbf{C}}$ (Logic/Discrete Math) Think of the contrapositive.

24. $\boxed{\textbf{D}}$ (Discrete Math)
$g(1) = 3g(2) = 3^2 g(3) = 9f(3) = 9 \cdot 2f(2) = 9 \cdot 2^2 f(1) = 9 \cdot 2^2 \cdot 1 = 36$.

25. $\boxed{\textbf{D}}$ (Number Theory) When in doubt, try numerical evidence like $x = -1, y = 2$ and $x = 0, y = 0$ to eliminate the bogus choices. More rigorously, note that $4x - 9y = 5(3x + 7y) - 11(x + 4y)$, and observe that each term is divisible by 11.

26. $\boxed{\textbf{E}}$ (Topology/Real Analysis) Since $\sin(\frac{1}{x}) \in [-1, 1]$, this is the only way to achieve disconnected subsets.

27. $\boxed{\textbf{A}}$ (Calculus) Since f is differentiable at $x = 1$, f is also continuous at $x = 1$. Using these two criteria around $x = 1$ (limitwise) yields $2a + b = 1$ and $a + b + c = 1$. Now, solve for (a, b, c).

28. $\boxed{\textbf{E}}$ (Calculus) This integral represents the positive area in a semicircle with radius $\frac{b-a}{2}$.

29. $\boxed{\textbf{C}}$ (Analytic Geometry) We are rescaling a semicircle in the y-direction, but not the x-direction.

30. $\boxed{\textbf{A}}$ (Calculus) Evaluate via substitution, obtaining $\frac{1}{2}[(f(b))^2 - (f(a))^2] = 0$, since $f(a) = f(b) = 0$.

31. $\boxed{\textbf{B}}$ (Calculus) Use L'Hopital's Rule.

32. $\boxed{\textbf{B}}$ (Linear Algebra) Use row reduction with a matrix whose row or columns are the given vectors; you should find that the resulting matrix has three pivots (and this equals the rank).

33. $\boxed{\textbf{E}}$ (Analytic Geometry) Note that $xy < 1$ represents the hyperbola (whose branches have the axes as their asymptotes).

34. $\boxed{\textbf{A}}$ (Geometry/Physics) Draw a picture, and divide the resulting triangle in two with the horizontal line from the bases of the mirror. Due to symmetry from the mirror, the two smaller triangles are congruent; hence $t - h = h$ (from the altitudes of the triangles).

35. $\boxed{\textbf{B}}$ (Linear Algebra) Use row reduction again; the bottom three rows are easily expressed in terms of the first two rows.

36. $\boxed{\textbf{A}}$ (Calculus) Minimize the square of the distance from the curve to the origin: $D = (x-0)^2 + (y-0)^2 = x^2 + (\frac{8}{x})^2 = x^2 + 64x^{-2}$. Setting $f'(x) = 0$ yields the critical points $x = \pm\sqrt{8}$, which yield the minimum distance $\sqrt{D} = 4$.

37. $\boxed{\textbf{D}}$ (Logic) A specific case need not imply the general case.

38. $\boxed{\textbf{A}}$ (Linear Algebra) Note that $M^3 = I$; so $M^{100} = M$.

39. $\boxed{\textbf{B}}$ (Calculus) The integrand is odd (and note that the point at $x = 0$ does not affect the value of the integral).

40. $\boxed{\textbf{C}}$ (Differential Equations) We can rewrite this as $d(xy) = xe^x dx$. Integrating both sides yields $xy = xe^x - e^x + C$. The initial condition shows that $C = 0$. Hence, we find that $y = e^x - \frac{e^x}{x}$, and $y(2) = \frac{e^2}{2}$.

41. $\boxed{\textbf{D}}$ (Calculus) Note that $y' = -e^{-x} + 1$ and $y'' = e^{-x}$. Evaluating these at $x = 1$, we find that y is increasing (with slope $1 - \frac{1}{e}$) and concave up at $x = 1$.

42. $\boxed{\textbf{D}}$ (Analytic Geometry) Choose two vectors on these rays; I am using $v = (1, 0, 1)$ and $w = (0, 1, 1)$. To find the angle θ between the rays, use the dot product formula $v \cdot w = ||v||||w|| \cos\theta$. This yields $1 = (\sqrt{2})(\sqrt{2}) \cos\theta$, and thus $\theta = \frac{\pi}{3}$.

43. $\boxed{\textbf{E}}$ (Algebra) Since $2 + i$ and $1 - i$ are roots of a polynomial with *real* coefficients, their conjugates are also roots. So the Factor Theorem yields
$f(x) = (x - (2+i))(x - (2-i))(x - (1-i))(x - (1+i)) =$
$((x-2)^2 + 1)((x-1)^2 + 1) = x^4 - 6x^3 + 15x^2 - 18x + 10$.

44. $\boxed{\textbf{B}}$ (Calculus) Use L'Hopital's Rule.
$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{h} = \lim_{h\to 0} \frac{f'(x_0+h) - (-f'(x_0-h))}{1} = 2f'(x_0).$$

45. $\boxed{\textbf{E}}$ (Calculus) Using the Ratio Test: $r = \lim\limits_{n\to\infty} \dfrac{e^{n+1}x^{n+1}/(n+1)!}{e^n x^n/n!} = \lim\limits_{n\to\infty} \dfrac{e|x|}{n+1} = 0$.
Since $r = 0 < 1$ for all x, this series has infinite radius of convergence.

46. $\boxed{\textbf{E}}$ (Algebra) Taking natural logarithms of both sides yields the identity $\log y \log x = \log x \log y$, which is true for all $x, y > 0$ (the domain for x and y due to their being arguments for the logarithms).

47. **D** (Discrete Math) Note that S has $2^8 = 256$ subsets; so $\mathcal{G}$ is only lacking 6 of them. Since there are 8 singleton subsets, $\mathcal{G}$ must include at least two of them (by the Pigeonhole Principle).

48. **D** (Linear Algebra) Note that $(ST)(p(x)) = S(xp(x)) = p(x) + xp'(x)$, and $(TS)(p(x)) = T(p'(x)) = xp'(x)$. Hence, $(ST - TS)(p(x)) = p(x)$ and so $ST - TS$ acts as the identity map of V onto itself.

49. **C** (Abstract Algebra) Since $x \neq x^{-1}$ for all $x \in G$, we have $x^2 \neq e$ for any $x \in G$. Thus, G has no elements of order 2. By Lagrange's Theorem, we conclude that $|G|$ is not divisible by 2. Moreover since G has a subgroup of order 7, $|G|$ is divisible by 7. The only choice fitting these two criteria is $|G| = 35$.

50. **D** (Probability) There are three possibilities: (i) A head by player 1 (and we're done, or a tail by player 1 followed by either a head or tail by player 2.

51. **E** (Calculus) Calling the limit L, letting $n \to \infty$ in the recurrence yields $L = \sqrt{3 + 2L}$. Solving for L (ignoring bogus negative solutions) yields $L = 3$.

52. **C** (Linear Algebra) Compute the eigenvalues from the characteristic equation: $\lambda^2 - 10\lambda + 24 = 0$ has roots (eigenvalues) $\lambda = 4, 6$.

53. **B** (Linear Algebra) V has dimension 4 (standard basis is $\{1, x, x^2, x^3\}$), and W has dimension 1, being generated by scalar multiples of the degree 3 polynomial $x(x-1)(x+1)$.

54. **E** (Abstract Algebra) $\phi(xy) = a(xy)a^2$, and $\phi(x)\phi(y) = (axa^2)(aya^2) = axa^3ya^2$. Since ϕ is a homomorphism, we need $\phi(xy) = \phi(x)\phi(y)$ and thus $a^3 = e$.

55. **C** (Calculus) Taking first partial derivatives yields $f_x = 3x^2 + 3y$ and $f_y = 3y^2 + x$. Setting these equal to 0, we obtain the critical points $(x, y) = (0, 0), (-1, -1)$. Next, we classify these with these with the Second Derivative Test. Note that $f_{xx} = 6x$, and $D = f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9$. Since $D(0, 0) < 0$, we have a saddle point at $(0, 0)$, and since $D(-1, -1) > 0$ and $f_{xx}(-1, -1) < 0$, we have a local maximum at $(-1, -1)$.

56. **A** (Calculus) The approximation given is the quadratic (Taylor) polynomial for $f(x) = \sqrt{x}$ at $x = 1$. Hence, the error at $x = 1.01$ given by $|\frac{f'''(c)(1.01-1)^3}{3!}|$ for some $c \in (1, 1.01)$. Since $|f'''(c)| = \frac{3}{8}c^{-5/2} < \frac{3}{8}$ on $(1, 1.01)$, the magnitude of the error is bounded above by $|\frac{3}{8}\frac{(0.01)^3}{3!}| = \frac{1}{16} \cdot 10^{-6}$. Finally, this error is positive, because the series is alternating, and the last term in the approximation is negative, leaving us an underestimate for $\sqrt{1.01}$.

57. **C** (Discrete Math) This is a Recurrence Relation set-up problem. To get this, ask yourself this question: To get to the k-th step, what are the possibilities of the previous step? In this case, we either tack on one digit, or a sign followed by a digit. These two possibilities (along with the enumerating) give the terms $10N_{k-1}$ and $4 \cdot 10N_{k-2}$, respectively.

58. **B** (Complex Analysis) If we write $f(z) = u(x, y) + iv(x, y)$ for real-valued u and v, then we know that $v(x, y) = 0$. The Cauchy-Riemann Equations immediately imply that $u(x, y)$ is constant.

59. **D** (Real Analysis) Note that this sum is measuring variation. We approach the supremum by taking a partition of $[0, 12]$ which consists of x-coordinates of local extrema. So, the upper bound appears to be $2 + 3 + 2 + 2 + 5 + 2 = 16$.

60. **C** (Probability) We can approximate this Binomial Distribution (with $n = 360$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$) with a Normal Distribution having mean $\mu = np = 60$ and standard deviation $\sigma = \sqrt{npq} = \sqrt{50} \approx 7$. So, 70 is between 1σ and 2σ from the μ. Since 68% of the data is within 1σ and 95% is within 2σ, and we're dealing with the right half of the normal distribution, we expect a probability within $(1 - 0.95)/2$ and $(1 - 0.68)/2$.

61. **C** (Linear Algebra) Since $A = A^{-1}$, we have $A^2 - I = 0$. Since A is 2×2, we conclude (by the Cayley-Hamilton Theorem) that $\lambda^2 - 1$ is the characteristic polynomial of A, with roots $\lambda = \pm 1$. So, the trace is the sum of the eigenvalues $1 + (-1) = 0$. Alternately, note that for instance $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ satisfies the criteria for the problem and has trace 0. (Sometimes, you can use specific examples to eliminate false answers!)

62. **B** (Calculus) By Green's Theorem, $\int_B (3ydx + 4xdy) = \iint_S (4 - 3)\, dA = Area(S) = 1$.

63. **D** (Calculus) Let $w = f(x)$, so that $x = f^{-1}(w)$ and $dx = (f^{-1}(w))'dw$. So, $\int_0^\infty f(x)\, dx = \int_1^0 w(f^{-1}(w))'\, dw$. Now, use integration by parts to obtain that $\int_0^\infty f(x)\, dx = 0 - 1f^{-1}(1) + \int_0^1 f^{-1}(w)\, dw = \int_0^1 f^{-1}(w)\, dw$. On the last step, note that $f^{-1}(1) = 0$.

64. **C** (Topology) The image of a compact set under a continuous map is compact.

65. **A** (Differential Equations) Rewrite as $(\frac{dy}{dx} + y)^2 = 0$. So, we have $\frac{dy}{dx} + y = 0$, which has general solution $y = Ce^{-x}$.

66. **B** (Abstract Algebra) Note that (III) is not closed under addition; although $1 + 0\sqrt{5}$ and $0 + 1\sqrt{5}$ are in the set, their sum $1 + 1\sqrt{5}$ does not satisfy the extra criterion, because $1^2 + 1^2 > 1$.