

## GRE Math Subject Test #2 Solutions.

1. **C** (Calculus) The Fundamental Theorem of Calculus directly gives  $F'(x) = \log x$ .
2. **D** (Discrete Math) Repeated use of the recursion yields  
$$F(n) = F(n-1) + \frac{1}{2} = F(n-2) + \frac{2}{2} = F(n-3) + \frac{3}{2} = \dots = F(n-(n-1)) + \frac{n-1}{2} = 2 + \frac{n-1}{2}.$$
Now, let  $n = 101$ .
3. **C** (Linear Algebra) Apply the formula for the inverse of a  $2 \times 2$  matrix.
4. **B** (Calculus) First, solve for  $b$  from evaluating the given integrals. Finding that  $b = \frac{3}{2}$ , the area equals  $\int_1^{3/2} (x^2 - x) dx = \frac{1}{6}$ .
5. **E** (Calculus) Remember that the derivative of a function at a point is the slope of the tangent line at the point. Accordingly, if  $f'(x) > 0$  (above the  $x$ -axis), then  $f(x)$  is increasing, and if  $f'(x) < 0$  (below the  $x$ -axis), then  $f(x)$  is decreasing. To narrow the remaining choice or two, measure some actual slopes of tangent lines from a potential plot of  $f$ .
6. **C** (Discrete Math) On each iteration of the loop,  $k = 999$  stays constant, while  $i$  doubles and  $p$  increases by 1. After 10 iterations of steps 2-4, we are at  $k = 999$ ,  $i = 2^{10} = 1024$ , and  $p = 10$ . Since  $i > k$  at last, we print 10.
7. **B** (Analytic Geometry) Consider the endpoints and orientation; this one is not bad.
8. **E** (Calculus) By  $u$ -substitution, this equals  $\frac{1}{2} \ln(x^2 + 1) \Big|_0^1 = \ln \sqrt{2}$ .
9. **A** (Discrete Math) Since  $f$  is 1-1 and mapping a finite set to itself, it is automatically a bijection. Therefore,  $f$  essentially permutes the elements of  $S$ ; and so there are  $n!$  possible functions  $f$ .
10. **B** (Real Analysis) This is similar to Dirichlet's function. Due to density of the rationals and irrationals in  $\mathbb{R}$ ,  $g$  is automatically discontinuous at any nonzero  $x$ . (The quickest way to establish this is to construct sequences of rationals and irrationals which converge to  $x$ , and note that applying  $f$  to each sequence gives different results.) However,  $g$  is continuous at  $x = 0$  by applying the Squeeze Theorem. For  $x > 0$ , we have  $1 \leq g(x) \leq e^x$ , and thus  $\lim_{x \rightarrow 0^+} g(x) = 1 = g(0)$ . Similarly, we have an analogous statement for  $x \rightarrow 0^-$ .
11. **A** (Algebra) Consider the cases  $x \geq y$  and  $x < y$  separately.
12. **C** (Real Analysis) Consider the definitions of supremum and limit point.
13. **A** (Probability) The probability of choosing the same color can be done in three pairwise disjoint cases (and add the results together):  $P(\text{blue}) = \frac{1}{C(8,2)}$ ,  
 $P(\text{red}) = \frac{C(4,2)}{C(8,2)}$ , and  $P(\text{yellow}) = \frac{1}{C(8,2)}$

14. **C** (Logic) Careful with the qualifiers!
15. **B** (Algebra) Since  $g(x) \in (a, x)$  and  $x > a$ , we see that  $g$  can not be constant. (Otherwise, use counterexamples to eliminate false answers.)
16. **D** (Linear Algebra) We solve  $A(0, 1, 1, 1) + B(0, 0, 0, 1) + C(1, 1, 2, 0) = (1, 2, m, 5)$  for  $A, B, C, D, m$ . Equating like entries, we find that  $A = C = 1$ , and thus  $m = 3$ .
17. **E** (Discrete Math) Write out the difference table, working downward and to the right, as needed. First, deduce that  $f(2) = 3$  and  $f(3) = 1$  in succession from the  $\Delta f$  column. Next, we find that  $\Delta^2 f(1) = -6$  and  $\Delta f(3) = 4$ . Finally,  $f(4) = 5$ .
18. **E** (Calculus) Draw the radii and label their lengths. We find that  $\frac{A(r)}{a(r)} = \frac{\pi(1^2 - r^2)}{\pi(1-r)^2} = \frac{1+r}{1-r}$ . Letting  $r \rightarrow 1^-$  yields  $\infty$  as the limit.
19. **B** (Abstract Algebra) Note that each row and column should have a permutation of all elements of the group; this eliminates III. As for why II is invalid, note that  $bc = d$  yields  $(bc)^2 = d^2$ , but  $(bc)^2 = b(cb)c = bdc = (bd)c = c^2 = a$ , but  $d^2 = b$ . (Observe that I is a multiplication table for the cyclic group of order 4.)
20. **D** (Calculus) I is true by the definition of the derivative, II is not true (try  $f(x) = x$ ), and III is true since  $f(x) = \frac{f(x)}{x} \cdot x$  (and apply the product rule for limits).
21. **D** (Calculus) First of all, we compute the equation of the tangent line at  $x = 0$  to be  $y = \frac{x}{2} + 1$ . Sketching the region now gives a right triangle with base and height having lengths 2 and 1; so the area equals  $\frac{1}{2} \cdot 2 \cdot 1 = 1$ .
22. **B** (Abstract Algebra) Note that the subset in (B) is not closed under inverses.
23. **C** (Geometry) Draw the figure, and note that  $\triangle OBA$  and  $\triangle COA$  are equilateral triangles. Hence,  $\angle BAC = 120^\circ$ .
24. **C** (Linear Algebra) Let  $(a, b, c, d)$  be orthogonal to both vectors. Then, the corresponding dot products equal 0:  $b + c + d = 0$  and  $a + b + c = 0$ . Solving this system yields the desired answer.
25. **D** (Linear Programming) The critical points occur at the vertices of the region; these are  $(\frac{1}{2}, \frac{1}{2})$ ,  $(1, 0)$ ,  $(2, 0)$ , and  $(2, 2)$ . Substituting these into  $f$ , we find that the maximum is 10 (and the minimum is  $\frac{1}{2}$ ).
26. **B** (Calculus) This is immediately seen from sketching the graph.
27. **E** (Calculus) Differentiating the relation yields  $f'(x) = -f'(1-x)$ . Now, let  $x = 0$ .
28. **C** (Linear Algebra) Use  $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ . This yields  $6 + 6 - \dim(V_1 \cap V_2) \leq 10$ , and so  $\dim(V_1 + V_2) \geq 2$ .

29. **B** (Calculus) Use integration by parts.  $\int_0^2 xp''(x) dx = (xp'(x) - p(x))\Big|_0^2 = -2$ .
30. **D** (Linear Algebra) The other choices are pretty clearly false!
31. **C** (Algebra) By the Rational Root Theorem, the only possible rational roots are of the form  $\frac{m}{n}$ , where  $m$  is a factor of  $b$  (the constant term), and  $n$  is a factor of 9 (the leading coefficient). So,  $\frac{1}{4}$  can not be a possible rational root.
32. **D** (Discrete Math) There are  $20!$  total arrangements, and either Pat is before or after Lynn (with both possibilities being equally likely).
33. **A** (Discrete Math) Via complements, this equals  $\lceil \frac{1000}{30} \rceil - \lceil \frac{1000}{\text{lcm}(30,16)} \rceil = 29$ .
34. **A** (Calculus) The first choice is true, because otherwise  $\lim_{x \rightarrow \infty} f(x)$  would not exist. [(B) is false because  $f'$  may not be differentiable, (C) and (D) are false, because  $f(x) = x$  satisfies the hypotheses, and for (E) use  $f(x) = e^{-x}$ .]
35. **B** (Calculus) At  $(x, y) = (0, \frac{\pi}{2})$ ,  $z = 1$ . Next, we evaluate the partial derivatives at  $(x, y) = (0, \frac{\pi}{2})$ :  $z_x = -1$  and  $z_y = 0$ . So, the equation of the tangent plane is  $z - 1 = -1(x - 0) + 0(y - \frac{\pi}{2})$ , or  $x + z = 1$ .
36. **D** (Probability) First, order the data (except  $x$ ). We find that there are three possibilities for the median:  $\eta(x) = 5, 7, x$ . Since  $\mu(x) = \frac{x+25}{5}$ , we have three possibilities:  $x = 0, 10, \frac{25}{4}$ , respectively for each value of  $\eta(x)$  (none of which are extraneous).
37. **B** (Calculus) Using the series for  $e$  (noting the index shifts below), this equals  $\sum_{k=1}^{\infty} \frac{k}{(k-1)!} = \sum_{k=1}^{\infty} \frac{(k-1)+1}{(k-1)!} = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = e + e = 2e$ .
38. **B** (Calculus) Direct computation.
39. **B** (Calculus) Note that  $f(x) = e^{-x}$  is decreasing; so use the sum with the biggest overestimate for  $\int_0^{10} e^{-x} dx$ .
40. **E** (Probability) This is a binomial trial/distribution question. Letting  $n$  denote the number of heads, we have  $P(n) = C(8, n)(1/2)^n(1/2)^{8-n}$ . So, this equals  $P(5) + P(6) + P(7) + P(8) = \frac{93}{256}$ .
41. **E** (Calculus) We find the critical point(s) by setting the first partial derivatives equal to 0. Since  $f_x = y - 3x^2$  and  $f_y = x - 3y^2$ , this yields  $(x, y) = (0, 0), (\frac{1}{3}, \frac{1}{3})$  as the critical points. Next, we classify them with the Second Derivative Test. Setting  $D = f_{xx}f_{yy} - (f_{xy})^2$ , since  $D(0, 0) < 0$ , we have a saddle point at  $(0, 0)$ , while  $D(\frac{1}{3}, \frac{1}{3}) > 0$  and  $f_{xx}(\frac{1}{3}, \frac{1}{3}) < 0$  implies that we have a local maximum at  $(\frac{1}{3}, \frac{1}{3})$ .
42. **D** (Analytic Geometry) Plotting the three points is instructive. Doing this, you should find that there are 3 possibilities for the fourth point.

43. **B** (Linear Algebra) Since  $(6, 7, 8)^t = -(0, 1, 2)^t + 2(3, 4, 5)^t$ , we see that  $A(6, 7, 8)^t = -A(0, 1, 2)^t + 2A(3, 4, 5)^t = -(1, 0, 0)^t + 2(0, 1, 0)^t = (-1, 2, 0)^t$ .
44. **D** (Calculus) Use logarithmic differentiation to compute  $f'(x)$ ; one finds that  $f'(x) = f(x) \cdot \frac{1}{2}(1 + \ln x)$ . Although  $f(x) > 0$ , note that  $1 + \ln x < 0$  (and thus  $f'(x) < 0$ ) for  $x \in (0, e^{-1})$ .
45. **A** (Calculus) When in doubt, check the units of the formulas.
46. **C** (Linear Algebra) Compute the eigenvalues of the matrix to be  $\lambda = \cos t \pm i \sin t$ . Since we want their sum to equal 1, this yields  $t = \frac{\pi}{3}$ .
47. **E** (Probability) Sketch the region inside the unit square  $[0, 1] \times [0, 1]$ ; the restriction  $|x - y| < \frac{1}{2}$  removes triangles with area  $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$  from the upper left and lower right corners of the square. So, the probability (area) equals  $1 - 2 \cdot \frac{1}{8} = \frac{3}{4}$ .
48. **A** (Calculus) See where the boundary curves map. Letting  $x = 0$ , we see that  $u = y, v = 1 + y$ , and thus  $v = u + 1$ . Letting  $x = 1$ , we see that  $u = y = v$ . Letting  $y = 0$ , we see that  $v = 1$ . Letting  $y = 1$ , we see that  $v = 2$ .
49. **E** (Calculus) These are all true; for I and III, use a  $u$ -substitutions ( $u = x - 3$  and  $u = 3x$ , respectively) to rewrite the right sides of each equality (to be) as their left sides. As for II, this is evident if you think of areas (or simply rewrite it as  $\int_a^3 f = \int_a^b f + \int_b^3 f$ ).
50. **D** (Calculus) There are four such functions:  $f(x) = \pm x, \pm|x|$ . That there are no others follows from  $f$  having to be continuous.
51. **D** (Calculus) Using the Ratio Test, we need  $r = |2x + y| < 1$ .
52. **D** (Linear Algebra) Being homogeneous,  $(0, 0, 0)$  is always a solution. Via row reduction, this system reduces to  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & b - 5 \end{bmatrix}$ . So, the system has one solution or infinitely many solutions whether  $b \neq 5$ .
53. **D** (Complex Analysis) Note that  $z = 1$  is the only singularity of the integrand inside  $C$ . So, rewriting this to apply Cauchy's Integral Formula, we obtain  $\int_C \frac{dz/(z+3)^2}{z-1} = 2\pi i \cdot \frac{1}{(z+3)^2} \Big|_{z=1} = \frac{\pi i}{8}$ . (Alternately, you can use the Residue Theorem, noting that the integrand has residue  $\frac{1}{16}$  at the simple pole  $z = 1$ .)
54. **A** (Differential Equations) Since  $V = 10^2 h = 100h$ , we obtain the differential equation  $V' + 0.25V = 1$ , which has solution  $V = Ce^{-0.25t} + 400$ . Regardless of the value of  $C$ , we conclude that  $\lim_{t \rightarrow \infty} V(t) = 400$ .
55. **D** (Calculus) Sketch the graphs of  $f'', f'$ , and then  $f$  in this order. Alternately, thinking of these graphs as acceleration, velocity, and position may be useful.

56. **E** (Topology/Real Analysis) (A) is not a metric, because it never equals 0. (C) is not a metric, because it equals 0 for all  $(x, y)$ . (B) and (D) fail to be metrics due to not satisfying the triangle inequality (try to find simple counterexamples). This leaves (E). With such questions, verifying the triangle inequality is the key. For (E), note that for all  $x, y, z \in S$ , we have  $d(x, z) \leq d(x, y) + d(y, z) \leq (\sqrt{d(x, y)} + \sqrt{d(y, z)})^2$ ; now take positive square roots of both sides.
57. **C** (Abstract Algebra) We eliminate II, because it is not closed under addition (note that  $x^2 + (-x^2 + x) = x$  is of odd degree). I and III are rings; check that they are closed under addition, additive inverses, and multiplication (and all contain 1).
58. **C** (Topology) II is false; suppose that  $f$  attains a maximum or minimum value on  $(0, 1)$ ; this would render  $S$  not open. I is true, since the continuous image of a connected set is connected, and III is true, because  $f([0, 1])$  is compact and thus bounded (and so its subset  $S = f((0, 1))$  is also bounded).
59. **A** (Abstract Algebra)  $x^5$  has order dividing 3, while the other two elements have order dividing 5; so we must have  $x^3 = x^9$  and  $x^3 = e$ , because the given set has exactly two elements (and not one). Finally, note that  $x^{13} = x$ , which is of order 3 by above.
60. **E** (Abstract Algebra) I is true, because  $s + s = (s + s)^2 = s^2 + 2ss + s^2 = 2(s^2 + s^2) = 0$ . Next III is true:  $s + t = (s + t)^2 = s^2 + st + ts + t^2 = s + st + ts + t$ . This implies that  $st + ts = 0$ , and thus  $st = ts$  by adding  $ts$  to both sides (and using I). Finally, II is true, because  $s + t = (s + t)^2 = s^2 + 2st + t^2 = s^2 + t^2$ , by using I and II.
61. **E** (Number Theory) Note that  $p^4 - 1 = (p + 1)(p - 1)(p^2 + 1)$ . Since  $p$  is odd, we know that among the first two factors, one is divisible by 2, and the other is divisible by 4. Moreover,  $p^2 + 1$  is even; so  $16 \mid (p^4 - 1)$  by the last few remarks. Next, note that by casework, we have  $3 \mid (p - 1)(p + 1) = p^2 - 1$  and  $5 \mid (p^2 - 1)$ . Thus,  $3 \cdot 5 \cdot 16 = 240 \mid (p^4 - 1)$ .
62. **A** (Algebra/Discrete Math) Ignoring higher degree terms,  $(1 + x)^3(2 + x^2)^{10} = (1 + 3x + 3x^2 + x^3)(2^{10} + 10 \cdot 2^9 x^2 + \dots)$ . So, the coefficient for  $x^3$  equals  $2^{10} + 30 \cdot 2^9 = 2^9(2 + 30) = 2^{14}$ .
63. **D** (Analytic Geometry/Calculus) The Intermediate Value Theorem applied to  $f(x) = x^{12} - 2^x$  shows the existence of roots in  $(-\infty, 0)$  and  $(1, 2)$ . On this latter interval, we have  $x^{12} > 2^x$ ; so there is a third root  $x > 2$ , because we know that  $2^x > x^{12}$  for sufficiently large  $x$ . This is it (and can be verified with Calculus as needed).
64. **C** (Real Analysis) Since  $f$  is continuous on a compact set, it is uniformly continuous on  $[0, 1]$ . Given  $\epsilon > 0$ , letting  $C = \epsilon$  and  $D = \delta$  yield I and II. Finally, III is false; try  $f(x) = x^{1/2}$ .

65. **A** (Calculus) By the Factor Theorem (calling the third zero  $k$ ), we have  $p(x) = (x + 3)(x - 2)(x - k) = x^3 + (1 - k)x^2 + (-k - 6)x + 6k$ . Now, differentiate and use  $p'(-3) < 0$  to find that  $k < -3$ .
66. **E** (Calculus) By Green's Theorem, this equals  $\iint (2x + 2) dA$ . Converting to polar coordinates yields  $\int_0^3 \int_0^{2\pi} (2r \cos \theta + 2) \cdot r d\theta dr = 18\pi$ .