GRE Math Subject Test #3 Solutions.

1. $\boxed{\textbf{D}}$ (Algebra) $f(g(x)) = g(x) + 3 = 5$ implies that $g(x) = 2$.

2. $\boxed{\textbf{C}}$ (Calculus) Direct substitution suffices (via continuity).

3. $\boxed{\textbf{A}}$ (Calculus) This equals $\frac{1}{2}e^{2x}\Big|_0^{\log 4} = \frac{15}{2}$.

4. $\boxed{\textbf{C}}$ (Discrete Math) We need $B \subseteq A$ via Venn diagrams or otherwise.

5. $\boxed{\textbf{A}}$ (Calculus) Since $x = -1 < 0$, we have $|x| = -x$. Now, differentiate.

6. $\boxed{\textbf{B}}$ (Calculus) By the Fund. Theorem of Calculus, $F'(b) = ((b+1)^2 + (b+1)) - (b^2 + b) = 2b + 2$. Setting this equal to 0 yields the critical point at $x = -1$. Since $F''(b) = 2 > 0$ for all b, we see that this yields a minimum.

7. $\boxed{\textbf{D}}$ (Calculus) Since we have $z > 0$ (as exponentials are never negative), the graph only appears in 4 of the octants.

8. $\boxed{\textbf{B}}$ (Calculus) Due to the radical, we need $\tan^2 x - 1 \geq 0$.

9. $\boxed{\textbf{E}}$ (Calculus) This equals $\frac{-1}{2}\log|2 - x^2|\Big|_0^1 = \frac{1}{2}\log 2$.

10. $\boxed{\textbf{A}}$ (Differential Equations) Integrating both sides, we obtain $f'(x) = f(x) + A$. Using $f'(0) = -1$, we find that $A = -1$. Now, we have $f'(x) - f(x) = -1$, a first order linear DE. Solving this yields $f(x) = Ce^x + 1$. Finally, $f(0) = 0$ yields $C = -1$.

11. $\boxed{\textbf{C}}$ (Calculus) Direct computation.

12. $\boxed{\textbf{E}}$ (Calculus) Using L'Hopital's Rule, we obtain $\lim_{x\to 0} \frac{2\cos(2x)}{1+\log(1+x)} = 2$.

13. $\boxed{\textbf{A}}$ (Calculus) This equals $\lim_{n\to\infty} \frac{x^{1-n}}{1-n}\Big|_1^n = \lim_{n\to\infty} \frac{n^{1-n}-1}{1-n} = 0$, because $\lim_{n\to\infty} n^{1-n} = \lim_{n\to\infty} \frac{1}{n^{n-1}} = 0$.

14. $\boxed{\textbf{D}}$ (Algebra) Solve $(\frac{1}{2})^n = \frac{1}{1000000}$ to find that $n = 6\log_2 10 = 6(\log_2 2 + \log_2 5) \approx 6(1 + 2.3) \approx 19$. Now, multiply this by 5 to find the number of years, which is approximately 100.

15. $\boxed{\textbf{D}}$ (Calculus) By the Fund. Theorem of Calculus, $f'(2) = \frac{1}{1+x^2}\Big|_{x=2} = \frac{1}{5}$. Via integration, $f(2) = \arctan 2 - \frac{\pi}{4}$.

16. $\boxed{\textbf{A}}$ (Calculus) Note that $\frac{h'}{h} = -g$; so $h(x) = Ce^{-g(x)}$. Hence, $f(x) = e^{g(x)}h(x) = C$.

17. $\boxed{\textbf{E}}$ (Trigonometry) Rewrite this as $\cos^2(\arccos\frac{\pi}{12}) = (\frac{\pi}{12})^2 = \frac{\pi^2}{144}$.

18. $\boxed{\textbf{D}}$ (Calculus) By geometric series, $f(x) = \frac{1}{1+x^2}$.

19. **A** (Differential Equations) This has characteristic equation $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$, which has $r = 1$ as a triple root. So, (A) is the answer.

20. **B** (Calculus) The curve of intersection is $8 = 2x^2 + 4y^2$, or $x^2 + 2y^2 = 4$. So, the region can be written as $y = -\sqrt{(4-x^2)/2}$ to $y = \sqrt{(4-x^2)/2}$ with $x \in [-2, 2]$. The integrand is $(6 - x^2 - 2y^2) - (-2 + x^2 + 2y^2) = 8 - 2x^2 - 4y^2$.

21. **D** (Calculus) Splitting the integral, we have $\int_0^a a^2\,dx + \int_a^1 ax\,dx = 1$. Solving for (real) a yields $a = 1$.

22. **E** (Abstract Algebra) $(b^2cb^4c^2)^{-1} = (c^2)^{-1}(b^4)^{-1}c^{-1}(b^2)^{-1} = c^{-2}b^{-4}c^{-1}b^{-2}$. Now, use $b^5 = c^3 = e$ to rewrite this with positive exponents to obtain cbc^2b^3.

23. **C** (Calculus) I is true by applying the Intermediate Value Theorem to $g(x) = f(x) - x$ on $[0, 1]$, and II is true by the Mean Value Theorem applied to $f(x)$ on $[0, 1]$. III is false (it is easy to sketch a counterexample).

24. **A** (Probability) A and B are independent iff $P(A \cap B) = P(A)P(B)$. The hypotheses of this problem imply that $P(A) = P(A)^2$ and thus $P(A) = 0$ or 1, giving a contradiction.

25. **D** (Real Analysis) (D) is true; consider $f(x) = \sqrt{x - \frac{1}{2}}$.

26. **C** (Linear Algebra) Let $v_2 = \langle a, b, c\rangle$. The vectors v_1 and v_2 are orthogonal iff their dot product equals 0: $a + b - c = 0$. Choice (C) satisfies this requirement.

27. **A** (Calculus) Think of its cross-sections.

28. **B** (Linear Algebra) I and III are certainly vector subspaces, being closed under addition and scalar multiplication. II is false, by taking A and B to be the x and y axes of $\mathbb{R}^2$, and noting that $A \cup B$ is not closed under addition. IV does not contain the zero vector.

29. **E** (Calculus) Although $\sum \frac{1}{2^k}$ converges, $\sum \frac{1}{k}$ diverges to ∞.

30. **E** (Calculus) If the subscripts get excessive, write this out for $n = 4$ or some special case.

31. **C** (Calculus) This equals $\int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 (x-1)\,dx$. The first integral equals $\frac{\pi}{4}$ most easily via area of a quarter unit circle.

32. **B** (Abstract Algebra) Check closure under addition, multiplication, and both inverses. I, II, and IV are not closed under multiplicative inverses.

33. **C** (Probability) The total number of arrangements of the apples is $n!$, while there is only one way to arrange the apples by increasing weight.

34. $\boxed{\textbf{E}}$ (Calculus) By the Fundamental Theorem of Calculus and Chain Rule, the derivative equals $e^{-(x^2)^2} \cdot \frac{d}{dx}x^2 = 2xe^{-x^4}$.

35. $\boxed{\textbf{B}}$ (Calculus) Note that this relation is an averaging relation. So, such f has constant slope. II and III fail, because $f(x) = -x$ is a counterexample.

36. $\boxed{\textbf{D}}$ (Algebra) $F(2,2) = F(F(1,2),1) = F(4,1) = 5$.

37. $\boxed{\textbf{B}}$ (Linear Algebra) The elementary row operation to transform the second row does not affect the value of the determinant, but rescaling the first row does. So, the determinant of the new matrix equals $-3 \cdot 9 = -27$.

38. $\boxed{\textbf{D}}$ (Calculus) This is a Riemann Sum for $\int_0^3 (x^2 - x)\, dx = \frac{9}{2}$.

39. $\boxed{\textbf{D}}$ (Algebra/Trig.) We need $1 + \sin(2\pi x) \leq 0$ iff $2\pi x = \frac{-\pi}{2} + 2\pi k$ for some integer k.

40. $\boxed{\textbf{A}}$ (Probability) This equals $\int_0^1 \int_0^1 (1 - yz)\, dy\, dx = \frac{3}{4}$.

41. $\boxed{\textbf{B}}$ (Linear Algebra) Rewrite this as $(A - I)X = 0$, which has solution $(a, 0)^t$ for any $a \in \mathbb{R}$.

42. $\boxed{\textbf{D}}$ (Calculus) By using Maclaurin polynomial, we have $f(x) = 1 + x + x^2 + E(x)$ for some error E. Since $f(1) < 5$, we see that $E(1) < 2$. Using the error bound for a second degree Taylor polynomial, we have $|E(1)| = |\frac{f'''(c)1^3}{3!}|$ for some $c \in (0,1)$. Combined with $E(1) < 2$, we find that $|f'''(c)| < 12$.

43. $\boxed{\textbf{E}}$ (Algebra) Let $f(x) = x^n - \sum_{i=0}^{n-1} a_i x^i$. This is continuous on $\mathbb{R}$, and I implies that $f(0) < 0$ while $f(1) > 0$, (while for III, it is vice versa). The result then follows from the Intermediate Value Theorem.

44. $\boxed{\textbf{D}}$ (Calculus) Applying L'Hopital's Rule two times yields $\lim_{h \to 0} \frac{3^2 P''(x+3h) + (-3)^2 P''(x-3h) - 0}{2} = 9P''(x)$.

45. $\boxed{\textbf{C}}$ (Algebra) We can uniquely solve for x^2 and y^3 via Cramer's Rule (since $ae \neq bd$). Solving for x and y yields at most two (real) possibilities and one (real) possibility, respectively. So, this system has at most two real solutions.

46. $\boxed{\textbf{E}}$ (Calculus) Although this can be done with only algebra, it's faster to use Taylor's Theorem, since $a_n = \frac{f^{(n)}(2)}{n!}$.

47. $\boxed{\textbf{C}}$ (Calculus) By Green's Theorem, this equals $\iint (1 - (-1))\, dA = 2\pi ab$, being twice the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

48. $\boxed{\textbf{D}}$ (Abstract Algebra) By the Chinese Remainder Theorem, $\mathbb{Z}_{ab} \cong \mathbb{Z}_a \times \mathbb{Z}_b$ iff $\gcd(a,b) = 1$ (and extends to more factors if the moduli are pairwise relatively prime). As such, the group in (D) can't be cyclic, because $\gcd(22,33) > 1$.

49. **D** (Probability) First, we find that $E[X] = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\, dx = 0$, and $E[X^2] = \int_{-1}^{1} x^2 \cdot \frac{3}{4}(1-x^2)\, dx = \frac{1}{5}$. Therefore, $\sigma = \sqrt{E[X^2] - (E[X])^2} = \frac{1}{\sqrt{5}}$.

50. **A** (Linear Algebra) The determinant's value will be linear x, y, z.

51. **B** (Abstract Algebra) Note that $\phi(1) = 1$, because $\phi(1) = \phi(1 \cdot 1) = \phi(1) \cdot \phi(1)$ and $\phi(1) \neq 0$ (since $\phi(0) = 0$ and we need ϕ to be 1-1). Since ϕ is additive, we see that $\phi(a) = a$ for any $a \in \mathbb{Z}$. Finally, using the fact that ϕ is multiplicative, this extends to all $a \in \mathbb{Q}$.

52. **D** (Linear Algebra) The matrix is determined by the images of the standard basis vectors: $(1,0) \mapsto (1,0) \mapsto (2,0)$, and $(0,1) \mapsto (0,-1) \mapsto (0,-2)$.

53. **A** (Complex Analysis) Since polynomials are entire, this integral equals 0 by Cauchy's Theorem.

54. **C** (Real Analysis) Apply Cauchy's Mean Value Theorem to f and g on $[0,1]$, and use $f'(x) \geq g'(x)$ for all $x \in [0,1]$.

55. **E** (Abstract Algebra) Subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some integer n.

56. **B** (Topology/Real Analysis) I and III are true. II is false; for instance let $X = \mathbb{R}$ with base $\{[a,b) : a, b \in \mathbb{R}\}$, and take $A = [0,1)$ and $B = [1,2)$. IV is false in the discrete topology (there are no limit points and all sets are open).

57. **C** (Discrete Math/Algorithms) This is a classic binary search algorithm.

58. **C** (Linear Algebra) The characteristic equation is $\lambda^2 - 6 = 0$.

59. **A** (Abstract Algebra) The subgroup H of order 12 is going to be a subgroup of the subgroup generated by H and K. So 12 must divide the order of this second subgroup, and note that 12 does not divide 30.

60. **D** (Set Theory) Drawing a Venn Diagram is useful for this one; ultimately we get 16 distinct sets in S (after just a few iterations).

61. **C** (Discrete Math) Put this in 1-1 correspondence to permutations of 5 E's and 7 N's.

62. **B** (Topology) I is false, since $[0, \frac{1}{2}] \cup \{[\frac{1}{2}, 1 - \frac{1}{n}) | n \in \mathbb{N}$ is a cover for $[0,1]$ which has no finite subcover. Using the same example, we see that III is false.

63. **D** (Calculus) Convert to polar coordinates: $\int_0^{2\pi} \int_0^2 e^{-r^2} \cdot r\, dr\, d\theta = \pi(1 - e^{-4})$.

64. **C** (Linear Algebra) We simply need solutions to $f'(x) = \lambda f(x)$.

65. **A** (Complex Analysis) Standard theorem.

66. **B** (Discrete Math) I and III are easily seen to be false. II is true by the Pigeonhole Principle.