GRE Math Subject Test #4 Solutions.

1. $\boxed{\textbf{B}}$ (Calculus/Geometry) This is simply the length of a semicircle with radius 1.

2. $\boxed{\textbf{E}}$ (Calculus) Note that $y' = 1 + e^x$. So, $m = y'(0) = 2$ and $(x, y) = (0, 1)$.

3. $\boxed{\textbf{D}}$ (Linear Algebra) Its dimension is at most 2.

4. $\boxed{\textbf{B}}$ (Analytic Geometry) Sketch the curves $y = e^x$ and $y = -x + 2$; they only intersect for some $x \in [0, 1]$.

5. $\boxed{\textbf{B}}$ (Algebra) Since $f(3) = 12 + 2b + 12 = 0$, we have $b = -12$. So, $f(5) = 27$.

6. $\boxed{\textbf{C}}$ (Analytic Geometry) Sketching the parabolas with the circles is the key.

7. $\boxed{\textbf{C}}$ (Calculus) $\int_{-3}^{3} |x + 1|\, dx = \int_{-3}^{-1} -(x + 1)\, dx + \int_{-1}^{3} (x + 1)\, dx = 10$.

8. $\boxed{\textbf{A}}$ (Analytic Geometry) Sketching the circle and (isosceles) triangle with common side having length 1, its area (via subdividing it into two congruent right triangles along the x-axis having angles θ with the positive x-axis) equals $2 \cdot \frac{1}{2} \cdot \cos\theta \sin\theta = \frac{1}{2}\sin(2\theta)$. Maximizing this area, we need $2\theta = \frac{\pi}{2}$ and so $\theta = \frac{\pi}{4}$.

9. $\boxed{\textbf{A}}$ (Calculus) Note that $\sqrt{1 - x^4} \leq \sqrt{1 - x^8} \leq 1 \leq \sqrt{1 + x^4}$ for $x \in [0, 1]$.

10. $\boxed{\textbf{B}}$ (Calculus) Note that $g'(2) = 0$, along with $g'(x) > 0$ for $x \to 2^-$, and $g'(x) < 0$ for $x \to 2^+$. So, g has a local maximum at $x = 2$. Similarly, g has a local minimum at $x = 5$.

11. $\boxed{\textbf{E}}$ (Algebra) $\sqrt{1.5} \cdot \sqrt{266} \cdot 266 = \sqrt{369} \cdot 266 \approx 19 \cdot 266 \approx 20 \cdot 265 = 5300$.

12. $\boxed{\textbf{C}}$ (Linear Algebra) One such matrix to use is $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; you can use such an example to try the potential eigenvectors. Here, $A(1, 1)^t = (3, 3)^t = 3(1, 1)^t$; so $(1, 1)^t$ is an eigenvector (with eigenvalue 3). The other choices are easily refuted, since $Av \neq \lambda v$ for some scalar λ.

13. $\boxed{\textbf{B}}$ (Algebra/Calculus) Draw a diagram of the yard; call the three sides l and w. Then, we have $2l + w = x$, and $A = lw$. Solving for w, we can rewrite the area as $A = -2l^2 + xl = -2(l - \frac{x}{4})^2 + \frac{x^2}{8}$, via completing the square. So, the maximal area equals $\frac{x^2}{8}$, being the vertex of a parabola pointing downward. (Alternately, you can set $A' = 0$ to find the critical point $l = \frac{x}{4}$ which yields the maximal area.)

14. $\boxed{\textbf{D}}$ (Number Theory) Computing mod 10 for the last digit yields $7^{25} \equiv (7^2)^{12} \cdot 7 \equiv (-1)^{12} \cdot 7 \equiv 7 \pmod{10}$.

15. $\boxed{\textbf{E}}$ (Calculus) Think of corner points or vertical tangents at $x = 0$.

16. $\boxed{\textbf{D}}$ (Calculus) Using the disk method, the volume equals $\int_0^\infty \pi(\frac{1}{\sqrt{1+x^2}})^2\,dx = \pi \arctan x\Big|_0^\infty = \frac{\pi^2}{2}$.

17. $\boxed{\textbf{B}}$ (Algebra/Calculus) Since the polynomial is of odd degree, it has at least one real root (by the Intermediate Value Theorem). However, since $f'(x) = 32x^4 + 8 > 0$ for all x, we see that f is increasing for all x. Hence, f has exactly one real root.

18. $\boxed{\textbf{A}}$ (Linear Algebra) The subspace in question is the null space of T. So the Rank-Nullity Theorem implies that it has dimension $6 - 4 = 2$.

19. $\boxed{\textbf{C}}$ (Calculus/Real Analysis) Use Cauchy's Mean Value Theorem.

20. $\boxed{\textbf{D}}$ (Real Analysis) This is reminiscent of Dirichlet's function. For any nonzero x, we can alternately use a sequence of rational/irrational numbers (via density of rationals/irrationals) which converges to a given nonzero real numbers. Applying f to these sequences will yield different limits. So, f is discontinuous for all $x \neq 0$. However, f is continuous at $x = 0$, which is most easily demonstrated with the Squeeze Theorem, because $\frac{|x|}{3} \leq |f(x)| \leq \frac{|x|}{2}$ for all $x \in \mathbb{R}$. Letting $x \to 0$ yields $\lim_{x\to 0} |f(x)| = 0$, and thus $\lim_{x\to 0} f(x) = 0$.

21. $\boxed{\textbf{C}}$ (Number Theory) Note that $2^2 \cdot 3 \cdot 5 \in P_{12} \cap P_{20}$.

22. $\boxed{\textbf{B}}$ (Linear Algebra) Note that 0 is not in the set in III; so it is automatically not a vector subspace. However, the other two are closed under addition and scalar multiplication.

23. $\boxed{\textbf{A}}$ (Calculus) Note that we want $y' = be^{bx} = 10$. Letting (x, y) denote the point of intersection between the curve and tangent line, we solve the system $y = 10x$ and $y = e^{bx} = \frac{10}{b}$. So, $10x = \frac{10}{b}$ yields $bx = 1$. From here, we find that $y = e^{bx} = e^1 = e$ and thus $b = \frac{10}{e}$.

24. $\boxed{\textbf{E}}$ (Calculus) Note that $h(x) = e^{x+x^2} - e^x$. Now, differentiate. (Alternately, use the product rule with the Fund. Theorem of Calculus.)

25. $\boxed{\textbf{A}}$ (Discrete Math) Repeated substitution yields $a_{30} = \frac{31}{29} \cdot \frac{30}{28} \cdot \ldots \cdot \frac{3}{2} \cdot a_1 = \frac{31\cdot 30}{2} \cdot 1 = 15 \cdot 31$.

26. $\boxed{\textbf{A}}$ (Calculus) Set the first partial derivatives $f_x = 2x - 2y$ and $f_y = -2x + 3y^2$ equal to 0. In particular, the critical points are on the line $y = x$.

27. $\boxed{\textbf{D}}$ (Linear Algebra) Solve these as a system of linear equations (row reducing via augmented matrices may be useful).

28. $\boxed{\textbf{D}}$ (Discrete Math) Can you see why 1, 2, or 3 are not sufficient?

29. $\boxed{\textbf{C}}$ (Calculus) Try $f(x) = x$ and $g(x) = x + 1$ for a quick counterexample.

30. $\boxed{\textbf{C}}$ (Logic) Be careful with the quantifiers.

31. **A** (Calculus) Remember that dy/dx measures the slope of a tangent at a particular point. The given differential equation implies that $dy/dx > 0$; hence, the solution must be increasing for all x. Moreover, since $\lim_{y\to\pm\infty}(y^4+1)=\infty$, the curves becomes nearly vertical when $|y|$ is large.

32. **D** (Abstract Algebra) I is not necessarily true if the multiplication is not commutative. On the other hand, II and III are checked to be true via induction.

33. **D** (Programming/Number Theory) The Euclidean Algorithm yields at the first two stages the following: $273 = 2(110) + 53$ and then $110 = 2(53) + 4$; the remainders are 53 and 4, respectively.

34. **E** (Analytic Geometry) This minimal distance is achieved by using the line segment connecting the centers (minus the lengths of the two radii).

35. **E** (Discrete Math) Treat the men as one unit (which can be arranges in 6! ways); so we can arrange 9 women and the group of men in 10! ways. Hence, the probability equals $\frac{6!\cdot 10!}{15!}$

36. **A** (Linear Algebra) Just because any two vectors in a set are linearly independent does not imply that the entire set is linearly independent. For instance, look at the set $\{(1,0,0),(0,1,0),(1,1,0)\}$.

37. **D** (Complex Variables/Algebra) Writing $z = x + iy$ and expanding yields $(x+iy)^2 = x^2 + y^2$. This simplifies to $2xyi - 2y^2 = y(2xi - 2y) = 0$. Since $x, y \in \mathbb{R}$, we conclude that $y = 0$.

38. **A** (Discrete Math/Set Theory) Let $x \in C$. Then, $f(x) \in f(C)$ and so $x \in f^{-1}(f(C))$. (The converse is not necessarily true if f is not 1-1.)

39. **B** (Calculus) By the Law of Cosines, $s^2 = r^2 - 2r\cos 110°$. Solving for $s - r$ yields $s - r = \frac{-2r\cos 110°}{s+r} = \frac{2r\cos 70°}{s+r}$. In particular, if the limit exists, then it is positive. Next, by the triangle inequality, we know that $r \le s + 1$. So, $s - r = \frac{2r\cos 70°}{s+r} \le \frac{2r\cos 70°}{r+(r-1)}$. Letting $r, s \to \infty$, we have that $\lim_{r,s\to\infty} s - r \le \cos 70° < 1$.

40. **C** (Abstract Algebra) Consider $f(x) = -x$ if $x \le 0$ and 0 if $x > 0$, and $g(x) = 0$ if $x \le 0$ and x if $x > 0$. Then, $(fg)(x) = 0$ for all $x \in \mathbb{R}$.

41. **E** (Calculus) By Green's Theorem, this line integral equals $\iint (1-(-1))\,dA = 2(\text{Area of the unit circle}) = 2\pi$.

42. **B** (Probability) Since X and Y are independent, $P(X = m \cap Y = n) = \frac{1}{2^{m+n}}$. So, $P(X > 3 \cup Y > 3) = 1 - P(X \le 3 \cap Y \le 3) =$
$1 - (\frac{1}{2^{1+1}} + \frac{1}{2^{2+2}} + \frac{1}{2^{3+3}} + \frac{2}{2^{1+2}} + \frac{2}{2^{1+3}} + \frac{2}{2^{2+3}}) = \frac{15}{64}$.
If you do not use complements, then you will need to use the geometric series.

43. $\boxed{\textbf{E}}$ (Algebra) Since z is a primitive fifth root of unity, we have $z^5 = 1$ and $z^4 + z^3 + z^2 + z + 1 = 0$. Repeated use of these facts reduces the expression to $5z^4 = 5e^{8\pi i/5} = -5e^{3\pi i/5}$.

44. $\boxed{\textbf{D}}$ (Probability) Since this is a binomial trial with $p = q = \frac{1}{2}$, all we need to do is compare binomial coefficients $C(100, k)$; the largest ones are those for which the values of k are near 50.

45. $\boxed{\textbf{D}}$ (Probability) In two words: Pigeonhole Principle.

46. $\boxed{\textbf{E}}$ (Abstract Algebra) All three choices yield homomorphisms.

47. $\boxed{\textbf{C}}$ (Calculus) The unit vector is $\frac{1}{\sqrt{2}}(-1, 0, 1)$. So, the work equals $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \frac{1}{\sqrt{2}}(-1, 0, 1) \cdot (1, 2t, 3t^2)\, dt = 0$.

48. $\boxed{\textbf{A}}$ (Calculus) The argument is valid.

49. $\boxed{\textbf{D}}$ (Abstract Algebra) We want the elements to all have order dividing 4. So, $\mathbb{Z}_{16}$ and $\mathbb{Z}_8 \times \mathbb{Z}_2$ are not possibilities, while the remaining three $\mathbb{Z}_4^2, \mathbb{Z}_4 \times \mathbb{Z}_2^2$, and $\mathbb{Z}_2^4$ are.

50. $\boxed{\textbf{B}}$ (Linear Algebra) $|A^2| = |A|^2 \geq 0$; so II is true. I is false (think of $I^2 = I$), and III is false (consider the diagonal matrix with diagonal entries 1 and -1; it square is I).

51. $\boxed{\textbf{B}}$ (Calculus) Due to the definition of the floor function, split ths integral at every integer value:
$\int_0^\infty \lfloor x \rfloor e^{-x}\, dx = \sum_{k=1}^\infty \int_k^{k+1} k \cdot e^{-x}\, dx = \sum_{k=1}^\infty k(e^{-k} - e^{-k-1}) = (1 - e^{-1}) \cdot \sum_{k=1}^\infty \frac{k}{e^k} = \frac{1}{e-1}$.
To evaluate this latter sum, differentiate the geometric series and then multiply both sides by x: $\frac{x}{(1-x)^2} = \sum_{k=1}^\infty kx^k$. Finally, let $x = \frac{1}{e}$.

52. $\boxed{\textbf{B}}$ (Topology/Real Analysis) If A is closed in $\mathbb{R}$, then it must contain its limit points, which is any element in $\mathbb{R}$ (as both the rational numbers and the irrational numbers are dense in $\mathbb{R}$).

53. $\boxed{\textbf{C}}$ (Calculus) First of all, $x + 4z$ has no critical points; so it suffices to minimize $f(x, y, z) = x + 4z$ subject to $g(x, y, z) = x^2 + y^2 + z^2 = 2$. By Lagrange Multipliers, $\nabla f = \lambda \nabla g$ yields $(1, 0, 4) = \lambda(2x, 2y, 2z)$. Solving this system yields the critical points $(\pm\sqrt{\frac{2}{17}}, 0, \pm 4\sqrt{\frac{2}{17}})$ yielding the extrema $\pm\sqrt{34}$.

54. $\boxed{\textbf{E}}$ (Geometry) Let r be the radius of the little circle in the second diagram, so the mid size circle has radius $r(1 + \sqrt{2})$. Computing the ratio of areas should now be straightforward.

55. $\boxed{\textbf{D}}$ (Discrete Math) Recall that $n!$ has $\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5^2} \rfloor + \ldots$ zeros in its decimal expansion. Check that 400! is the first factorial ending in 99 zeros (399! ends in 98 zeros), and 404! is the last such factorial. So, there are 5 possibilities.

56. **E** (Topology/Real Analysis) Note that the function in (E) does not satisfy the triangle inequality: $(x-0)^2 + (y-0)^2 \leq (x-y)^2$ is not generally true.

57. **E** (Calculus) Simply apply the Ratio Test, and note that $r = \frac{1}{e} < 1$. The fact that $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$ will be useful.

58. **E** (Linear Algebra) All three claims are true. Writing $A = PBP^{-1}$ for some P, we have $A - 2I = PBP^{-1} - 2I = P(B-2I)P^{-1}$; so I is true.
Next, II is true, because $trA = tr(PB)P^{-1} = tr(BP)P^{-1} = trB$.
Finally, III is true, because $A^{-1} = (P^{-1})^{-1}B^{-1}P^{-1} = PB^{-1}P^{-1}$.

59. **A** (Complex Analysis) Using the Cauchy-Riemann Equations, we find that $g_y = u_x = 2$ and $g_x = -u_y = 3$. Hence $g(x,y) = -3x + 2y + C$. Using $g(2,3) = 1$, we find that $C = 1$.

60. **E** (Abstract Algebra) Examining this shape, we see that it has the reflections and rotations that a regular pentagon enjoys; hence its symmetry group is D_5 of order 10.

61. **C** (Set Theory) Think of power sets.

62. **D** (Topology) III is false, because $[0,1] \cup [2,3]$ is compact, but not connected. I and II are familiar properties of compact sets.

63. **D** (Calculus) For $x \neq 0$, setting $f'(x) = 0$ yields 2 critical points. However, we have a third one at $x = 0$, since $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$.

64. **D** (Real Analysis) The limiting function f equals 0 if $x = 0$ and 1 otherwise; hence we have pointwise convergence. Since f is not continuous on $[0,1]$, the convergence for $f_n(x)$ is not uniform on $[0,1]$. The final assertion III is true by direct computation.

65. **B** (Real Analysis/Topology) I is true; consider $f(x) = 0$ if $x \in (0, 1/3]$, $3(x - 1/3)$ if $x \in [1/3, 2/3]$, and 1 otherwise. II is false, since the continuous image of a compact set is compact. As for III, note that f is either strictly increasing or decreasing since f is a bijection. Assume that f is increasing (the other case is treated similarly), we know that $f(a) = 1$ for some $a \in (0,1)$. Hence $f(b) > 1$ for some $a < b < 1$, which is a contradiction.

66. **B** (Abstract Algebra) I is false, since R could be non-commutative. II is true, because since $\{0\}$ and R are the only ideals, each element has its own multiplicative inverse. III is false, by considering $R = \mathbb{Z}_2$.