GRE Math Subject Test #5 Solutions.

1. **E** (Calculus) Apply L'Hôpital's Rule two times:
$\lim_{x\to 0} \frac{\cos(3x)-1}{x^2} = \lim_{x\to 0} \frac{-3\sin(3x)}{2x} = \lim_{x\to 0} \frac{-9\cos(3x)}{2} = -\frac{9}{2}$.

2. **C** (Geometry) Note that a line segment connecting the center of the circle and the vertex of the equilateral triangle bisects the interior angle of the triangle. Next, drawing a perpendicular bisector to one of the sides of the bisected angle (which is also a radius of the circle), we obtain a $30° - 60° - 90°$ triangle whose legs are the radius (length 2) and half the side length of the original equilateral triangle (length $\frac{s}{2}$). Hence, $\tan 30° = \frac{2}{s/2}$ and thus $s = 4\sqrt{3}$. Finally, the area of the equilateral triangle equals $\frac{\sqrt{3}}{4}s^2 = 12\sqrt{3}$.

3. **D** (Calculus) Using the substitution $u = \log x$, the integral becomes
$\int_{-3}^{-2} \frac{du}{u} = \log|u||_{-3}^{-2} = \log(\frac{2}{3})$.

4. **A** (Linear Algebra) Since $\dim(V+W) = \dim V + \dim W - \dim(V \cap W)$, we have $\dim(V \cap W) = 8 - \dim(V+W)$. However, since $\dim(V+W) \leq 7$, it follows that $\dim(V \cap W) \geq 8 - 7 = 1$.

5. **E** (Probability) Since 1,4, and 9 are the only possible squares, we can't count $(1,1), (2,4), (4,2), (3,9)$, and $(9,3)$. So, the probability equals $\frac{10^2-5}{10^2} = 0.95$.

6. **C** (Algebra) Note that $2^{1/2} = 2^{3/6} = 8^{1/6}$ and $3^{1/3} = 9^{1/6}$.

7. **C** (Calculus) Remember that f is increasing when $f' > 0$, and f is decreasing when $f' < 0$.

8. **B** (Abstract Algebra) Just know the axioms of a group; in this case, the nonzero real integers are not closed under multiplication.

9. **A** (Calculus) Know the geometric significance of the first and second derivatives.

10. **A** (Algebra/Analytic Geometry) Simplifying yields $y = 3x + 1$.

11. **B** (Calculus) Applying the Cylindrical Shell method yields the volume
$\int_0^1 2\pi x(x - x^2)\, dx = \frac{\pi}{6}$.

12. **B** (Abstract Algebra) Recall that the only groups of prime order are cyclic. There are at least two nonisomorphic groups of even order greater than 2 (cyclic and dihedral), and there are two nonisomorphic groups of order 9 ($\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$).

13. **D** (Calculus) Apply the Mean Value Theorem to f on $[0,3]$: $\frac{f(3)-f(0)}{3-0} = f'(x) \geq -1$ for some $x \in (0,3)$. So, $f(0) \leq -8$.

14. **B** (Calculus) Let $x = c$, obtaining $3c^5 + 96 = 0$ and thus $c = -2$.

15. **A** (Set Theory) Suppose there are $a, b \in S$ such that $f(a) = f(b)$. Apply g to both sides: $(g \circ f)(a) = (g \circ f)(b)$. Since $g \circ f$ is one-to-one, we have $a = b$. (If you want to eliminate B, C, D, use $f(x) = e^x$ and $g(x) = x^2$.)

16. **B** (Logic) Via contraposition, if C is false, then A and B are both true or both false. Checking truth tables now yields the conclusion.

17. **B** (Calculus) First, $x^3 = 10 - x$ has one real solution, because $f(x) = x^3 + x - 10$ is increasing for all real x (check the derivative). Next, $x^2 + 5x - 7 = x + 8$ is easily checked to have two real solutions, while $7x + 5 = 1 - 3x$ has one real solution. The remaining two possibilities have zero and one real solution, respectively (by graphing the individual functions manually).

18. **A** (Calculus) Differentiating term by term yields the geometric series.

19. **E** (Complex Variables) We get different limits to $z = 0$ by trying the paths (i) $y = 0$ with $x \to 0$ (yielding 1) and (ii) $y = x$ with $x \to 0$ (yielding -1).

20. **E** (Calculus) Noting that $g(0) = e$ so that $g(e) = g(g(0))$, this limit represents $\frac{d}{dx} g(g(x))|_{x=0} = g'(g(0))g'(0) = g'(e)g'(0) = 2e^{2e+1} \cdot 2e = 4e^{2e+2}$.

21. **B** (Calculus) Note that the second term of the integrand is odd, while the first term is even. So, the integral reduces to $2\int_0^{\pi/4} \cos t \, dt = \sqrt{2}$.

22. **C** (Calculus) The volume equals $\int_{-1}^{1} \int_{x^2}^{2-x^2} (y + 3)\, dy\, dx = \frac{32}{3}$.

23. **D** (Abstract Algebra) Note that 1 is not in $\{0, 2, 4, 6, 8\}$.

24. **E** (Linear Algebra) If you're clever, you don't need to explicitly solve this system of linear equations! Check that $(-5, 1, 1, 0)$ is a solution. Since this system is homogeneous with a nontrivial solution, it has infinitely many solutions. Being linear, sums and scalar multiples of solutions yield other solutions. Hence, choice (E) must be false, as the other four choices were shown to be true. (In fact, the solution set is spanned by two linearly independent vectors!)

25. **A** (Calculus) Since we are given the derivative of the graph of the *derivative* of h, to find when h has a point of inflection (when h'' changes sign), we simply need to find an interval where h' changes from increasing to decreasing or vice versa.

26. **D** (Number Theory) Note that $3x \equiv 5 \equiv -6 \mod 11$ yields $x \equiv -2 \mod 11$, while $2y \equiv 7 \equiv 18 \mod 11$ yields $y \equiv 9 \mod 11$. So, $x + y \equiv -2 + 9 \equiv 7 \mod 11$.

27. **D** (Algebra) Since $(1 + i)^2 = 2i$, we have $(1 + i)^{10} = (2i)^5 = 32i$.

28. **D** (Calculus) From the equation of the tangent line, $f(1) = 4$ and $f'(1) = 3$. So, $(g \circ f)'(1) = g'(f(1))f'(1) = g'(4)f'(1) = \frac{1}{2\sqrt{4}} \cdot 3 = \frac{3}{4}$.

29. **C** (Graph Theory) Draw them!

30. **A** (Calculus) This is true iff $y = \log x$ and $y = cx^4$ share a point of tangency (and a tangent line) for some $x > 0$. Setting their derivatives equal yields $\frac{1}{x} = 4cx^3$ and thus $cx^4 = \frac{1}{4}$. Then, $\log x = \frac{1}{4}$ which implies that $(x, y) = (e^{1/4}, \frac{1}{4})$. Solving for c now yields $c = \frac{1}{4e}$.

31. **C** (Linear Algebra) Recall that λ is an eigenvalue of a matrix A iff $A - \lambda I$ is not an invertible matrix. In particular, a matrix is not invertible if it has two rows that are the same; this shows that 2 is an eigenvalue. On the other hand, 3 is not an eigenvalue, because row reducing $A - 3I$ gives a matrix with pivots in each column. (Alternately, you can solve the characteristic equation $|A - \lambda I| = 0$ for the eigenvalues.)

32. **E** (Calculus) Apply the Fundamental Theorem of Calculus with the Chain Rule (you may want to split the integral with a constant bound to facilitate this).

33. **C** (Calculus) Write out a few derivatives for $(x - 1)e^{-x}$ and spot a pattern (note the sign changes between the even and odd number derivatives)!

34. **B** (Linear Algebra) Choices A, C, and D are equivalent (and true), while choice E is easy to verify. However, B is false, because $Ax = x$ is equivalent to $(A - I)x = 0$ and $A - I$ is not invertible (as its first column are all zeros and thus $\det(A - I) = 0$).

35. **B** (Calculus/Linear Algebra) We can do this with Calculus methods, but a quicker approach is to use the normal vector to the plane, namely $\langle 2, 1, 3\rangle$. The desired point on the plane comes from finding where the line passing through the origin with direction vector $\langle 2, 1, 3\rangle$ (namely $\mathbf{r}(t) = (2t, t, 3t)$) intersects the plane. Substituting this into the equation of the plane yields $t = \frac{3}{14}$. Hence, the desired point is $\mathbf{r}(\frac{3}{14}) = (\frac{3}{7}, \frac{3}{14}, \frac{9}{14})$.

36. **C** (Real Analysis/Topology) Verify the definition of an open set in $\mathbb{R}$.

37. **C** (Linear Algebra) Note that the zero map satisfies $P^2 = P$ but is not invertible; hence I is false. Similarly, III is false, because P could be a reflection map. As for II, note that the minimal polynomial of P divides $t^2 - t$ yielding possible eigenvalues 0 and 1. Checking that (nontrivial) Jordan blocks are not equal to their square, we conclude that P must be diagonal.

38. **C** (Geometry) Suppose there were at least 4 acute angles in the convex decagon. As the sum of the interior angles of the decagon equals $(10 - 2) \cdot 180° = 1440°$, the obtuse angles account for at least $1440° - 4 \cdot 89° = 1084°$. Since there are at most 6 non-acute angles, there is at least angle measuring more than $180°$, since the average among these 6 angles measures $\frac{1084°}{6} > 180°$. This contradicts the decagon being convex. As one can explicitly construct a convex decagon with 3 acute angles, we are done.

39. **D** (Algorithms) The first integer to be printed is 2 (since i gets replaced with 2 in the bigger while loop).

40. **C** (Abstract Algebra) Remember that composition of two functions is not commutative and that functions are generally not linear!

41. **A** (Analytic Geometry/Calculus) The equation of the line l is given by $\mathbf{r}(t) = (t, -1, 4-t) = (0, -1, 0) + t(1, 0, -1)$. Hence, the normal vector to the desired plane is given by $\langle 1, 0, -1\rangle$. Since the plane passes through the origin, its equation is given by $x - z = 0$.

42. **E** (Topology) Check the definitions of an open/closed sets in a metric space.

43. **A** (Calculus) Since $x'(t) = 2t + 2$ and $y'(t) = 12t^3 + 12t^2$, we have $\frac{dy}{dx} = \frac{12t^3+12t^2}{2t+2} = 6t^2$. So, $\frac{d^2y}{dx^2} = \frac{(6t^2)'}{2t+2} = \frac{6t}{t+1}$. At $t = 2$ (which gives the point $(8, 80)$), we have $\frac{d^2y}{dx^2} = 4$.

44. **B** (Differential Equations) This is a first order linear DE. Multiplying both sides by the integrating factor gives us $(e^{x^2/2}y)' = xe^{x^2/2}$. Integrating both sides and solving for y yields $y = 1 + Ce^{-x^2/2}$. (Using $y(0) = -1$, we can find that $C = -1$, although that is not necessary to answer this question.) Hence, $\lim_{x\to-\infty} y(x) = 1 + 0 = 1$.

45. **C** (Analytic Geometry/Trigonometry) Any such solutions must occur when $x \in [0, 1]$, since the two curves will never intersect for $x > 1$. Since the period of $\cos(97x)$ equals $\frac{2\pi}{97}$, this function on $[0, 1]$ repeats itself $(\frac{2\pi}{97})^{-1} \approx 15.43$ times. On each period, the two curves intersect twice, and they intersect one more time on the "0.43" part. Hence, they intersect 31 times.

46. **C** This is a related rates problem. Let x denote the distance the ladder is moving away from the wall, and y denote how far above the ground the latter is. Then, the ladder, the wall, and the ground form a right triangle, giving us $x^2 + y^2 = 9^2$. Differentiating both sides with respect to time t yields $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $y = 3$, we have $x = 6\sqrt{2}$. This combined with $\frac{dx}{dt} = 2$ yields $\frac{dy}{dt} = -4\sqrt{2}$ meters per second.

47. **B** (Real Analysis) This is reminiscent of Dirichlet's function. For any nonzero x, we can alternately use a sequence of rational/irrational numbers (via density of rationals/irrationals) which converges to a given nonzero real numbers. Applying f to these sequences will yield different limits. So, f is discontinuous for all $x \neq 0$. However, f is continuous at $x = 0$, which is most easily demonstrated with the Squeeze Theorem, because $-5x^2 \leq |f(x)| \leq 3x^2$ for all $x \in \mathbb{R}$. Letting $x \to 0$ yields $\lim_{x\to0} f(x) = 0 = f(0)$. Hence, f is continuous only at $x = 0$. Next, f is differentiable at $x = 0$, because $f'(0) = \lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{f(x)}{x} = 0$. (The last equality follows from applying the Squeeze Theorem to $3|x| \leq |\frac{f(x)}{x}| \leq 5|x|$.)

48. **B** (Calculus) Remembering to normalize the direction vector, the directional derivative equals $\nabla g(0, 0, \pi) \cdot \frac{1}{\sqrt{14}}\langle 1, 2, 3\rangle = \frac{3}{\sqrt{14}}$.

49. **B** (Abstract Algebra) Such an element in cycle notation is of the form $(abc)(de)$; this element has order $\mathrm{lcm}(3, 2) = 6$.

50. **D** (Abstract Algebra) In order for II to be true, the ideal must be *generated* by products of elements from U and V.

51. **E** (Linear Algebra) Row reduction shows that the first and third columns span the column space of out matrix. Remember that an orthonormal basis consists of orthogonal vectors (dot products of distinct vectors equal 0) each having length 1; this eliminates D as an answer (although it is a basis for the column space). However, we can write both $(1,-1,2)^T$ and $(1,1,0)^T$ in terms of the vectors in (D); normalizing these vectors gives us E as the answer.

52. **A** (Combinatorics) There are 20! ways to arrange the 20 courses. However, since each of the 10 professors teach 2 of the courses, we have over counted by a factor of 2^{10}.

53. **A** (Calculus) Differentiating under the integral sign yields $g'(x) = -\int_0^x f(y)\,dy$. So, $g''(x) = -f(x)$ by the Fundamental Theorem of Calculus. In order for g to be thrice continuously differentiable, we need f to be at least one time continuously differentiable. Since we don't know whether g is four times continuously differentiable, we can't guarantee that $f'(x)$ is continuously differentiable.

54. **C** (Calculus/Probability) The area of the rectangular region is $3 \cdot 4 = 12$ units. The region where $x \geq y$ is a triangle whose area is $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$. Hence, the desired probability equals $1 - \frac{9/2}{12} = \frac{5}{8}$.

55. **E** (Calculus) Note that the integrand can be rewritten as $\frac{1}{1+e^{bx}} - \frac{1}{1+e^{ax}}$. Since $\int_0^\infty \frac{dx}{1+e^{nx}} = \int_0^\infty \frac{e^{-nx}\,dx}{1+e^{-nx}} = \frac{1}{n}\log 2$, the answer now immediately follows.

56. **D** (Calculus/Algorithms) Note that II is false, because $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$, which is cubic. We can see that I is true by examining the graphs of $y = \log x$ and $y = \sqrt{x}$. Finally for III, note that via power series, $|\sin x - x| = |\frac{1}{6}x^3 - ...|$, and the series on the right side is alternating.

57. **C** (Real Analysis) By the Squeeze Theorem, I is true. To see that II is not true, consider $f(x) = \sin(\frac{1}{x})$. Note that we can choose x_n so that $f(x_n)$ oscillates infinitely often between -1 and 1; hence, $\{f(x_n)\}$ is divergent and therefore not Cauchy. Finally, III is true: Since $\{x_n\}$ is a convergent sequence in $\mathbb{R}$, it is Cauchy. So, for any $\delta > 0$, there exists a positive integer N such that $|x_m - x_n| < \delta$ for all $m, n > N$. Since g is uniformly continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that for all x_m, x_n such that $|x_m - x_n| < \delta$, we have $|g(x_m) - g(x_n)| < \epsilon$. Hence, there exists a positive integer N such that $|g(x_m) - g(x_n)| < \epsilon$ for all $m, n > N$. So, $\{g(x_n)\}$ is Cauchy and thus convergent.

58. **B** (Calculus) The arc length $L(\theta)$ of the helix on $[0, \theta]$ is given by $\int_0^\theta \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt = \theta\sqrt{26}$. Moreover, $D(\theta) = \sqrt{25 + \theta^2}$. Since $L(\theta_0) = 26$, we find that $\theta_0 = \sqrt{26}$ and thus $D(\theta_0) = \sqrt{51}$.

59. **E** (Linear Algebra) Note that A, B, and D are equivalent to A being invertible. Condition C shows a way to find the inverse of A. Finally, a counterexample for E follows from letting A be the 3×3 matrix with 1 in the upper left corner and 0s elsewhere, $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, and $v_3 = (1,0,0)$.

60. **D** (Real Analysis) As x gets farther from 1, $f(x)$ gets corresponding farther away from $f(1)$.

61. **E** (Differential Equations) Let x denote the amount of salt (in grams) in the tank. Since the concentration of salt is $\frac{x}{100}$ g/L, the salt is leaving the tank at a rate of $\frac{4x}{100}$ g/min. At the same time, salt is entering the tank at a rate of $4(0.02) = 0.08$ g/min. So, the differential equation for this problem is $\frac{dx}{dt} = 0.08 - 0.04x$. Solving this equation yields $x(t) = 2 + Ce^{-0.04t}$ for some constant C. Since $x(0) = 3$, we find that $C = 1$. Finally, $x(100) = 2 + e^{-4}$ grams.

62. **C** (Topology) By density of the rationals and irrationals in $\mathbb{R}$, we see that S is neither open nor closed (and thus not compact). Finally, D is false, because any vertical line segment in S with fixed x-coordinate being irrational is completely contained in S (as a connected segment). In fact, S is path-connected (using vertical and horizontal segments with the constant component being irrational.

63. **E** (Real Analysis) Note that $|\inf(A)|$ could possibly be greater than $|\sup(A)|$.

64. **E** (Calculus) This can be done with a straightforward surface integral calculation. To speed this up, we closed off the surface by including the circular base of the hemisphere on the plane $z = 0$. (Note that the flux on the base B equals 0, since $\iint_B \mathbf{F} \cdot d\mathbf{S} = \iint \langle x, y, 0\rangle \cdot \langle 0, 0, 1\rangle \, dA = 0$.) Hence, the flux in question equals the flux on the closed upper hemisphere, which we can compute with the Divergence Theorem, yielding $\iiint 3 \, dV = 3 \cdot (\frac{1}{2} \cdot \frac{4\pi}{3}) = 2\pi$.

65. **E** (Complex Variables) Use the Cauchy-Riemann Equations. Since $u(x, y) = e^x \sin y$ and $v(x, y) = g(x, y)$, $u_x = v_y$ yields $g_y = e^x \sin y$ and $u_y = -v_x$ yields $g_x = -e^x \cos y$. Integrating these equations yields $g(x, y) = -e^x \cos y + C$ for some constant C. Computing $g(3, 2) - g(1, 2)$ is now easy to do.

66. **B** (Abstract Algebra/Number Theory) Since 17 is prime, $\mathbb{Z}_{17}^{\times}$ is a cyclic group with $17 - 1 = 16$ elements. Thus, a generator of this group is an element whose order is 16. (By Lagrange's Theorem, it suffices to check powers of the element to proper factors of 16.) Observe that 16 has order 2, because $16^2 \equiv (-1)^2 \equiv 1 \bmod 17$, while 8 has order 8, because $8^8 \equiv 64^4 \equiv (-4)^4 \equiv (-1)^2 \equiv 1 \bmod 17$. However 5 has order 16, because $5^8 \not\equiv 1 \bmod 17$.