Exponential Sums and the Multisection Formula

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Outline:

- Exponential Sums
- The Multisection Formula
- Examples and Related Results
- Parting Shots

Exponential Sums

- I'll refer to any series of the form

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \ldots$$

  as an **exponential sum**.

- In particular, consider special subseries of

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Motivating Problem

- (Stewart, Calculus, 3rd Edition, p.680, #15) Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \ldots$$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \ldots$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \ldots$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$.

Solution (sketch):

- Note that these series are derivatives of one another: $u' = w$, $v' = u$, and $w' = v$.

- Letting $F = u^3 + v^3 + w^3 - 3uvw$, we see that $dF/dx = 0$. Hence F is constant.

- Using $u(0) = 1$, $v(0) = 0$, and $w(0) = 0$, we see that this constant equals 1.

More interesting question: Can we find closed forms for u, v, and w?

Feasibility: We know that

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

A solution via Differential Equations

- u satisfies the linear differential equation $u''' = u$.

- This DE has characteristic equation $r^3 - 1 = 0$, with solutions $r = 1, \dfrac{-1 \pm \sqrt{-3}}{2}$.

- **Note:** These are all cube roots of unity! Let $\omega = \dfrac{-1 + \sqrt{-3}}{2}$. Then, $r = 1, \omega, \omega^2$.

- General solution: $u = C_1 e^x + C_2 e^{\omega x} + C_3 e^{\omega^2 x}$.

DE solution, continued

- Find C_1, C_2, and C_3 through the initial conditions $u(0) = 1$, $u'(0) = 0$, and $u''(0) = 0$.

- Solving, $C_1 = C_2 = C_3 = \dfrac{1}{3}$.

- Therefore, $u = \dfrac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$.

Conclusion:

$$u = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}.$$

$$v = u'' = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!} = \frac{e^x + \omega^2 e^{\omega x} + \omega e^{\omega^2 x}}{3}.$$

$$w = u' = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!} = \frac{e^x + \omega e^{\omega x} + \omega^2 e^{\omega^2 x}}{3}.$$

Observations:

- Note the similarities between these series' closed forms and those for hyperbolic sine and cosine.

- This would have been lost if we had used real numbers instead of complex numbers.

  For example,

$$u = \frac{e^x + 2e^{-x/2}\cos(\sqrt{3}x/2)}{3}.$$

- Moral:
  Complex numbers often simplify matters!

A generalisation:

Let ω denote a primitive n-th root of unity (such as $e^{2\pi i/n}$), and fix $l \in \mathbb{N}$ where $0 \leq l < n$.

$$\sum_{k=0}^{\infty} \frac{x^{nk+l}}{(nk+l)!} = \frac{e^x + \omega^{-l} e^{\omega x} + ... + \omega^{-l(n-1)} e^{\omega^{n-1} x}}{n}.$$

In particular,

$$\sum_{k=0}^{\infty} \frac{x^{nk}}{(nk)!} = \frac{e^x + e^{\omega x} + e^{\omega^2 x} + ... + e^{\omega^{n-1} x}}{n}.$$

Remark: Note that the above series sum over l to e^x. (Compare to $e^x = \cosh x + \sinh x$.)

Alternate proof (sketch).

- We didn't need to use DE's to solve the previous problem.

- Since ω satisfies $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1) = 0$, we see that $\omega^{n-1} + \omega^{n-2} + ... + \omega + 1 = 0$.

- In fact, for any $s \in \mathbb{N}$,

$$(\omega^s)^{n-1} + (\omega^s)^{n-2} + ... + (\omega^s) + 1 = \begin{cases} 0 \text{ if } s \nmid n \\ n \text{ if } s \mid n. \end{cases}$$

- Using this last fact, the proof is a straightforward computation.

What about other series, besides that of e^x?

Using $(\omega^s)^{n-1} + (\omega^s)^{n-2} + ... + (\omega^s) + 1 = 0$ whenever s is not a multiple of n, we may easily deduce the following theorem.

Multisection Formula:
Let $\omega = e^{2\pi i/n}$ and $l \in \mathbb{N}$ such that $0 \leq l < n$.
If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$\sum_{j=0}^{\infty} a_{nj+l} \, x^{nj+l} = \frac{1}{n} \sum_{s=0}^{n-1} \omega^{-ls} f(\omega^s x).$$

Remark: Note that the above series sum over l to $f(x)$ (and hence its name).

First published by Thomas Simpson (1759):

"The invention of a general method for determining the sum of every second, third, fourth, or fifth, etc. terms of a series taken in order the sum of the whole series being known",

Philosophical Transactions of the Royal Society of London, Vol. 50, pp 757-769.

Special case $m = 2$:

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$\sum_{j=0}^{\infty} a_{2j}\, x^{2j} = \frac{1}{2}[f(x) + f(-x)]$$

$$\sum_{j=0}^{\infty} a_{2j+1}\, x^{2j+1} = \frac{1}{2}[f(x) - f(-x)]$$

This is essentially the classic result that a function may be written in terms of an even function and an odd function!

An example

- $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ with $x = 1$ yields

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

- This, with letting $x = -1$, yields

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... = 2^{n-1}$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ... = 2^{n-1}$$

- What is

$$\sum_{k \equiv 0(3)} \binom{n}{k} = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + ...?$$

An example (continued)

- Applying the Multisection Formula to $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ with $0 \le k < m$ and $x = 1$:

$$\sum_{k\equiv 0(3)} \binom{n}{k} = \frac{1}{3} \sum_{s=0}^{2} (1+\omega^s)^n.$$

- Since $\omega^2 = \overline{\omega}$, applying De Moivre's Theorem yields

$$\sum_{k\equiv 0(3)} \binom{n}{k} = \frac{1}{3}\left[2^n + 2\cos\left(\frac{n\pi}{3}\right)\right].$$

An example (concluded)

In general, we get the following result
(C. Ramus, 1834)

$$\sum_{k \equiv r(m)} \binom{n}{k} = \frac{1}{m} \sum_{j=0}^{m-1} \left(2 \cos \left(\frac{j\pi}{m} \right) \right)^n \cos \frac{j(n-2r)\pi}{m}$$

Related problems

- Consider **alternating exponential sums** of the form

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{nk+l}}{(nk+l)!}.$$

- When $n = 1$, this is e^{-x}.

- When $n = 2$, these are $\cos x$ ($l = 0$) and $\sin x$ ($l = 1$).

- What about $n > 2$?

Alternating exponential sums (continued)

- Let ω denote a primitive $(2n)$-th root of unity, and fix $l \in \mathbb{N}$ where $0 \leq l < n$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{nk+l}}{(nk+l)!} = \frac{1}{n} \sum_{s=0}^{n-1} \omega^{-l(2s+1)} e^{\omega^{2s+1} x}.$$

  In particular,

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{nk}}{(nk)!} = \frac{e^{\omega x} + e^{\omega^3 x} + ... + e^{\omega^{2n-1} x}}{n}.$$

- Let $s_l(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{nk+l}}{(nk+l)!}$.

  Then, $e^{\omega x} = \sum_{l=0}^{n-1} \omega^l \, s_l(x)$.

  In the case when $n = 2$, this reduces to Euler's identity $e^{ix} = \cos x + i \sin x$.

Parting shots

- Generalise the Multisection Formula to polynomials/series in more than one variable.

- Develop an analogue of the Multisection Formula for $\sum_{j=0}^{\infty} (-1)^j a_{nj+l}\, x^{nj+l}$.

- What happens if we use other groups to act on a variable instead of the n-th roots of unity (i.e, the cyclic group of order n) in the Multisection Formula?