THE BENJAMIN-ONO EQUATION IN WEIGHTED SOBOLEV SPACES

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PROBLEM

Consider the IVP associated to the Benjamin-Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

with $\mathcal{H}$ denoting the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}\,\text{p.v.}\left(\frac{1}{x} * f\right)(x) = -i\,(\text{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x).$$

CONSERVED QUANTITIES

Real valued solutions of the IVP (2) satisfy infinitely many conservation laws including the following three:

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \quad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx,$$

$$I_3(u) = \int_{-\infty}^{\infty} \left(|D^{1/2}u|^2 - \frac{u^3}{3}\right)(x,t)dx,$$

where $D = \mathcal{H}\partial_x$. This would allow one to deduce global well-posedness results from local well-posedness results.

TIME EVOLUTION OF MOMENTS

Proposition 1. *For $u(x,t)$ a solution of (2)*

(a)

$$\frac{d}{dt}\int xu(x,t)dx = \frac{1}{2}I_2(u),$$

(b)

$$\frac{d}{dt}\int xu^2(x,t)dx = 2I_3(u),$$

(c)

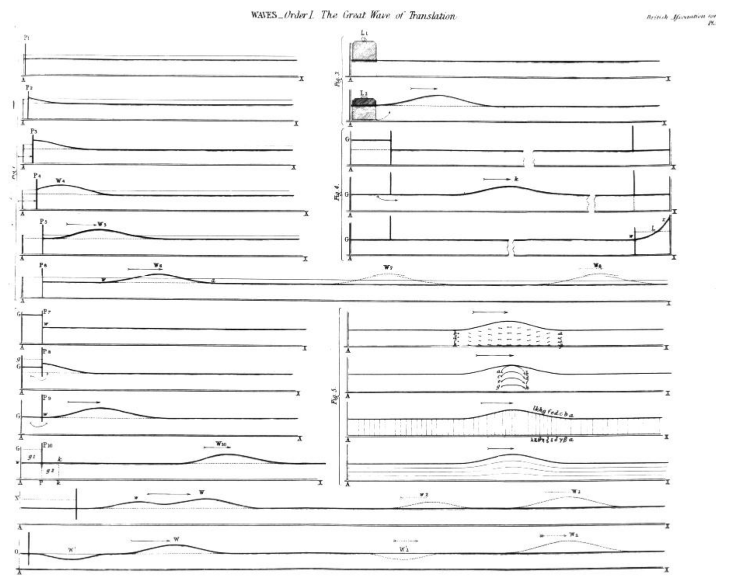
$$\frac{d}{dt}\int x^2 u(x,t)dx = \int xu^2(x,t)dx.$$

We shall combine these identities with a good cancelation between the linear and nonlinear terms in the Duhamel formula below for solutions of (2) to obtain a proof of Theorem 1. Theorem 2 builds on this and an extra condition avoiding the estimate of any additional moments.

Duhamel:

$$\widehat{u}(\xi,t) = e^{it|\xi|\xi}\widehat{u_0}(\xi) - \int_0^t e^{i(t-t')|\xi|\xi}\,\widehat{u\partial_x u}(\xi,t')dt'.$$

TRAVELING WAVE



We seek $u(x,t) = \phi(x-t)$ satisfying the Benjamin-Ono equation, i.e.,

$$-\phi + \mathcal{H}\phi' + \left(\frac{\phi^2}{2}\right)' = 0.$$

Integrate and apply Fourier transform:

$$-\widehat{\phi} + 2\pi|\xi|\widehat{\phi} + \frac{1}{2}\widehat{\phi} * \widehat{\phi} = 0.$$

After some computation, we have that:

$$\phi(x) = \frac{4}{1+x^2}.$$

KNOWN RESULTS

PROBLEM: minimal regularity in Sobolev scale

$$H^s(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) \,:\, (1+|\xi|^2)^{s/2}\widehat{f} \in L^2(\mathbb{R})\},$$

where $s \in \mathbb{R}$, which guarantees that the IVP for the BO is locally wellposed (LWP) i.e. existence and uniqueness hold in a space embedded in $C([0,T] : H^s(\mathbb{R}))$ AND the map data-solution from $H^s(\mathbb{R}) \longrightarrow C([0,T] : H^s(\mathbb{R}))$ is locally continuous.

1. Kato, Iorio, Abdelouhab et. al. $s > 3/2$
2. Ponce: $s \geqslant 3/2$
3. Koch-Tzvetkov: $s > 5/4$
4. Kenig-Koenig: $s > 9/8$
5. Tao: $s \geqslant 1$
6. Burq-Planchon: $s > 1/4$
7. Ionescu-Kenig: $s \geqslant 0$.

Molinet-Tzvetkov-Saut: for any $s \in \mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ is not C^3!! It cannot be solve by using only a contraction principle argument!!!

MAIN RESULT

GOAL: to study real valued solutions of the IVP for the BO in weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \qquad s, r \in \mathbb{R}. \tag{3}$$

$$\dot{Z}_{s,r} = Z_{s,r} \cap \{\widehat{f}(0) = 0\}$$

and uniqueness properties of solutions in these spaces.

Theorem A. *(Fonseca-Ponce 2011)*

(i) Suppose $u \in C([0,T] : Z_{2,2})$ is a solution of the IVP (2) such that at two different times, $t_1, t_2 \in [0,T]$

$$u(\cdot, t_j) \in Z_{5/2,5/2} \qquad j = 1, 2,$$

then $\hat{u_0}(0) = 0$.

(ii) Suppose $u \in C([0,T] : \dot{Z}_{3,3})$ is a solution of the IVP (2) such that at three different times, $t_1, t_2, t_3 \in [0,T]$

$$u(\cdot, t_j) \in \dot{Z}_{7/2,7/2} \qquad j = 1, 2, 3,$$

then $u(x,t) \equiv 0$.

It is known that we cannot have uniqueness at two different times with $r = 7/2$. *Can we have uniqueness at two different times by strengthening the decay assumption?* The answer is **no** if we increase to $r = 4$. For instance, if $t_1 \neq t_2$ and $u(t_1), u(t_2) \in \dot{Z}_{5,4}$, there exists solutions of the BO $u \neq 0$.

Can $r = 4$ be improved? The answer is **yes** as the next theorems show.

Theorem 1. *There exists $u_0 \in \dot{Z}_{7,5}$ such that $u_0 \neq 0$ and $\exists! t^* \neq 0$ such that the corresponding solution*

$$u(t^*) \in \dot{Z}_{7,5},$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2}\int_{\mathbb{R}} xu_0(x)dx.$$

Theorem 2. *The above is still true with $r = 5 + 1/2^-$ instead of $r = 5$.*

REFERENCES

[1] G. Fonseca and G. Ponce, (2011) *The IVP for the Benjamin-Ono equation in weighted Sobolev spaces*, J. Func. Anal **260** 436-459.

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NEW DIRECTION: THE DISPERSION GENERALIZED BO EQUATION

Considered here is the IVP for the DGBO:

$$\begin{cases} \partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{2}$$

with values $0 < a < 1$.

1. Do solutions to DGBO share similar decay properties with BO?
2. Is there a way to formulate a contraction principle in weighted Sobolev spaces?

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