Lower Bounds for Radio Numbers of Graph Products and Radio Numbers of Products of Paths

Shera Fournier[1], Frank Garcia[2], Adriana Juarez[3], Carissa Mendoza[4], Michael Nava[5]

[1]Loyola Marymount University, [2]Pierce College, [3]Concordia University, [4]University of Redlands, [5]California State University Channel Islands

1. Definitions and Examples

- The **distance** between any two vertices u, v of a graph G, $d(u,v)$, is the length of the shortest path between u and v. The **diameter** of a graph G, $\mathrm{diam}(G)$, is the maximum distance between a pair of vertices in G.
- A **radio labeling** is a function $c: V(G) \to Z_+$ that satisfies the following **radio condition** for all distinct vertices u and v:

$$d(u,v) + |c(u) - c(v)| \geq \mathrm{diam}(G) + 1.$$

The **span of** c, $span(c)$, is the maximum value of c.

- The **radio number of a graph** G, $rn(G)$, is the minimum span taken over all radio labelings of G.

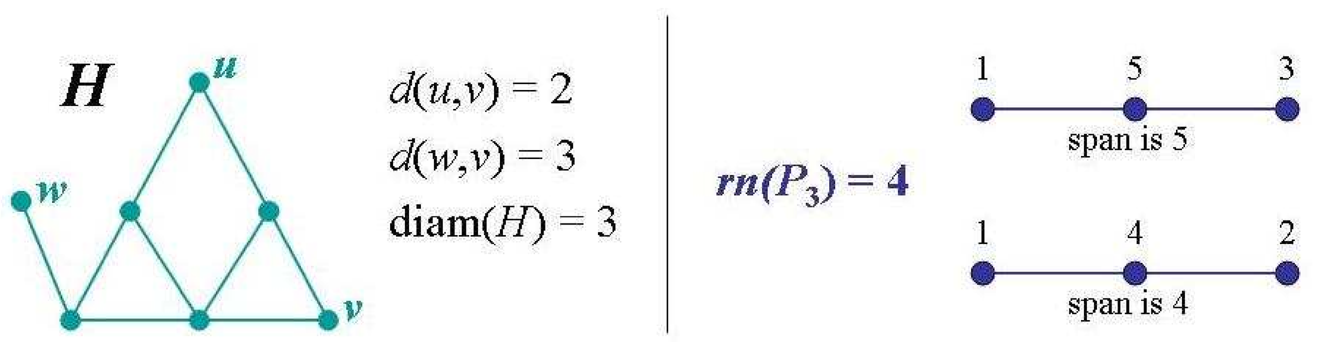

Figura 1: *Examples Illustrating Key Definitions*

- The **Cartesian product** of two graphs G and H, $G\square H$, is a graph with vertex set $V(G\square H) = V(G) \times V(H)$ and edges $E(G\square H) = \{(g,h),(g',h')\}$ where $g = g'$ and (h,h') is an edge in H, or $h = h'$ and (g,g') is an edge in G.

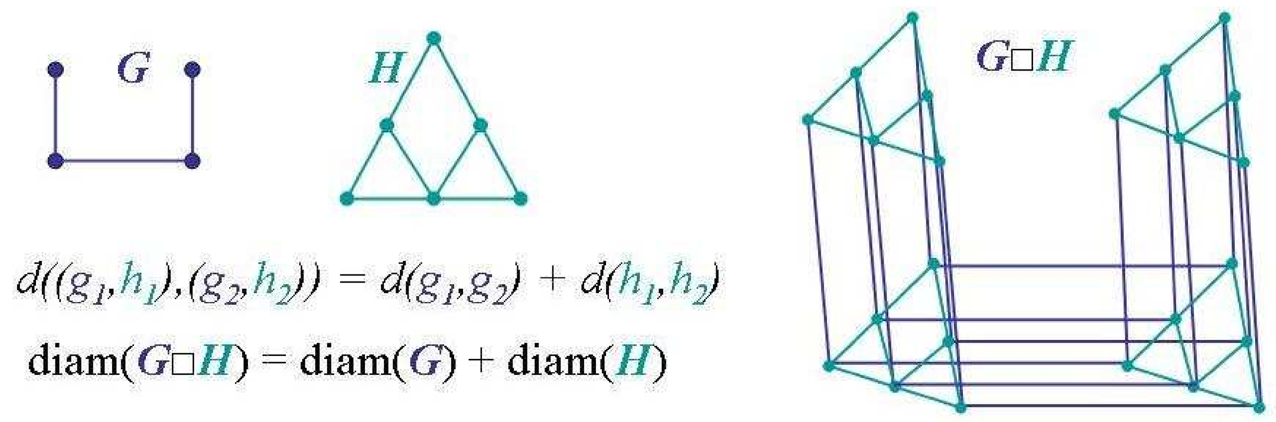

Figura 2: *Example of a Graph Cartesian Product*

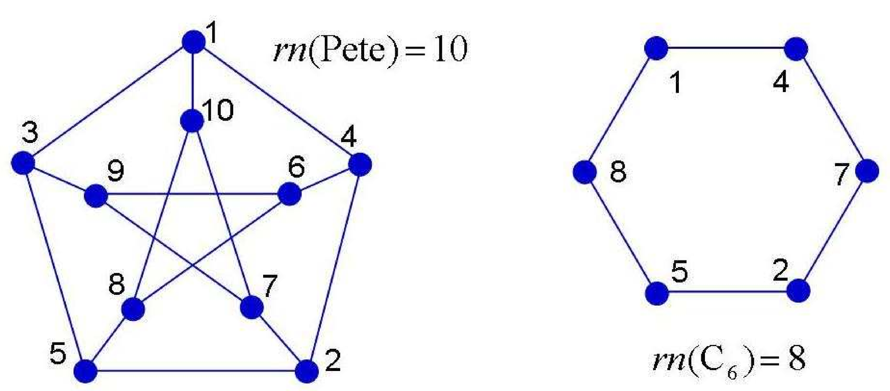

Figura 3: *Radio Labelings of the Peterson Graph and the 6-Cycle*

2. Lower Bounds for the Radio Numbers of Product Graphs

Lema: Let c be a radio labeling of G and let $\{x_1, .., x_n\}$ be the vertices of G arranged so that $c(x_i) < c(x_j)$ whenever $i < j$. Let $\phi(G) = max\{c(x_{i+2}) - c(x_i)\}$. Then

$$\phi(G) \geq \left\lceil \frac{3 + 3\mathrm{diam}(G) - D_G}{2} \right\rceil.$$

Teorema:

- Radio Numbers: $rn(G\square H) \geq rn(G) + rn(H) - 2$
- Vertices: $rn(G\square H) \geq |V(G\square H)| = |V(G)| \cdot |V(H)|$
- Gap Function: $rn(G\square H) \geq 2 + \left(\left\lfloor \frac{|V(G\square H)|}{2} \right\rfloor - 1\right)(\phi(G) + \phi(H) - 2)$

3. Analysis

Pregunta: How do the different lower bounds compare to one another and to the actual radio numbers (when known)?

Product	Vertices	Gap Function	Radio Numbers	Actual
$C_4\square P_2$	8	-	5	11
$C_n\square P_2$	$2n$	-	$\sim \frac{n^2}{8}$	$\sim \left(\frac{n^2}{4} + \frac{7n}{4}\right)$
$C_4\square C_4$	16	30	8	30
$C_n\square C_n$	n^2	$\sim \frac{n^3}{4}$	$\left(\sim \frac{n^2}{8} + \sim \frac{n^2}{8}\right)$	$\sim \left(\frac{n^3}{4} + \frac{n^2}{8}\right)$
$P_4\square P_4$	16	30	10	≤ 51
$P_{100}\square P_{100}$	10000	499902	9800	?
$P_n\square P_n$	n^2	$\sim \frac{n^3}{2}$	$(\sim \frac{n^2}{2}) + (\sim \frac{n^2}{2})$	?
$P\square P$	100	100	18	100

4. Products of Paths

Teorema: $rn(P_n\square P_n) \leq n^3 - n^2 + n - 1$ when n is even.

Represent $P_n\square P_n$ with an $n \times n$ grid: squares correspond to vertices; an edge shared by two squares corresponds to an edge in the graph.

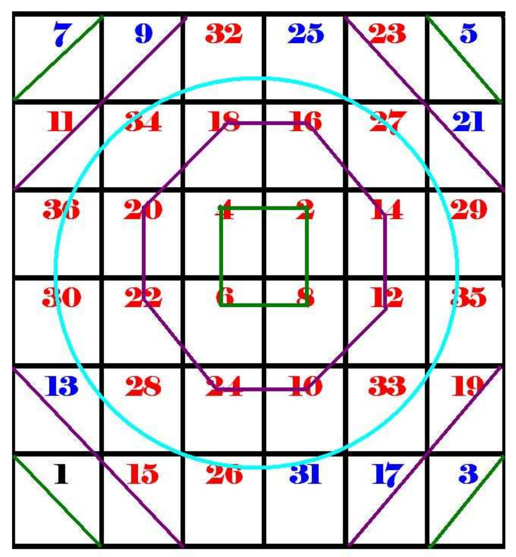

Figura 4: *Order of Labels, n Even*

Proof Sketch

Labeling Algorithm: Vertex 1 is assigned the label 1. If a vertex' order is indicated in red, add n to the previous label value to obtain this vertex' label, otherwise add $n-1$. We prove this produces a radio labeling.

Determining the Span:

- There are $\left(\frac{n}{2} - 1\right)$ "diagonal stages" with 4 "blue vertices" in each stage, so we add $4\left(\frac{n}{2} - 1\right)n$.
- There is one "circular stage" with 2 "blue vertices" so we add $2n$.
- We subtract the number of blue vertices from n^2 to obtain the number of "red vertices": add $\left(n^2 - 4\left(\frac{n}{2} - 1\right) - 2\right)(n-1)$.
- Correct for the first label (counted as red): add $-n+1$.

Teorema: $rn(P_n\square P_n) \leq \dfrac{2n^3 - 2n^2 - n + 3}{2}$ when n is odd.

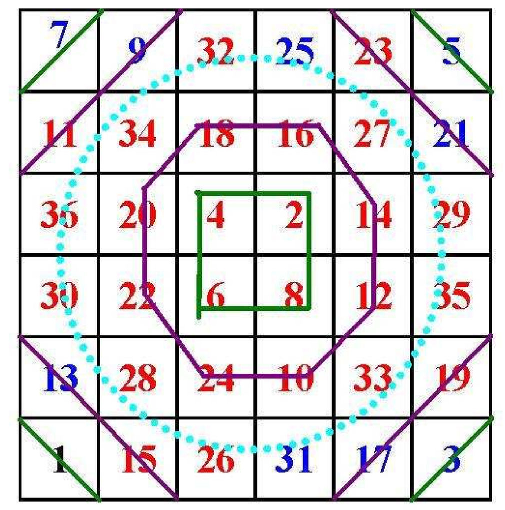

Figura 5: *Order of Labels, n Odd*

The proof strategy is similar to the strategy used when n is even.

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References

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