Radio Numbers of Graph Products: Upper Bounds

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1. Context and Definitions

- The **distance** between any two vertices u, v of a graph G, $d(u,v)$, is the length of the shortest path between u and v. The **diameter** of a graph G, $\text{diam}(G)$, is the maximum distance between a pair of vertices in G.
- A **radio labeling** is a function $c : V(G) \to Z_+$ that satisfies the following **radio condition** for all distinct vertices u and v:
$$d(u,v) + |c(u) - c(v)| \geq \text{diam}(G) + 1.$$
The **span of** c, $span(c)$, is the maximum value of c.
- The **radio number of a graph** G, $rn(G)$, is the minimum span taken over all radio labelings of G.

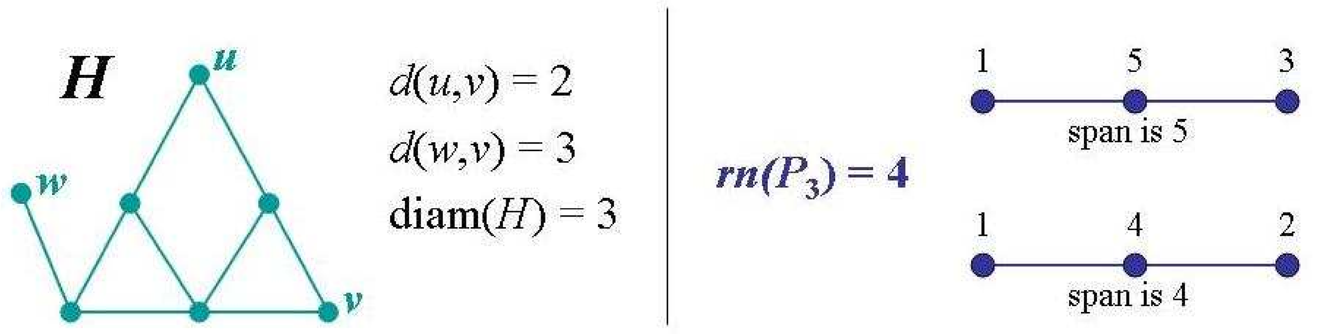

Figura 1: *Examples Illustrating Key Definitions*

- The **Cartesian product** of two graphs G and H, $G \square H$, is a graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edges $E(G \square H) = \{(g,h),(g',h')\}$ where $g = g'$ and (h,h') is an edge in H, or $h = h'$ and (g,g') is an edge in G.

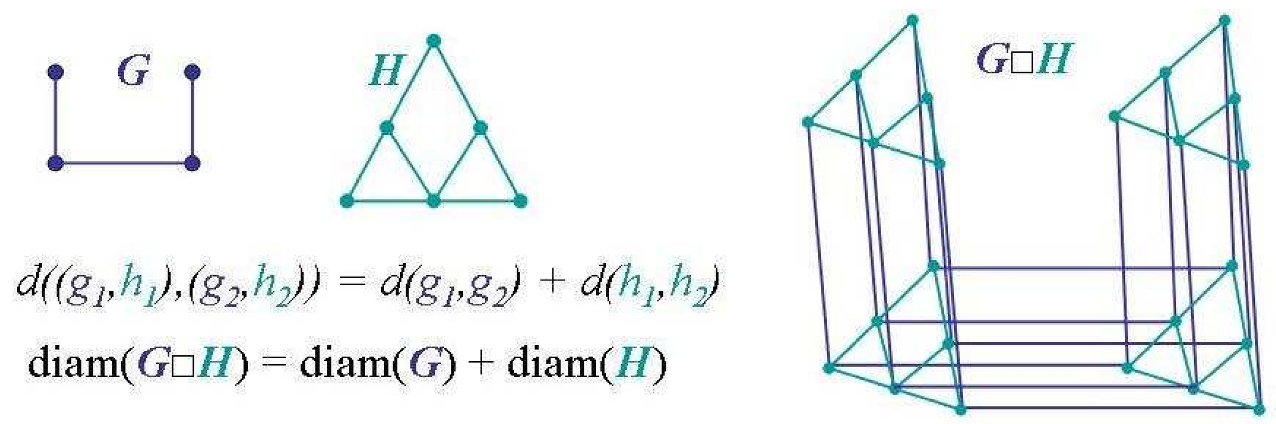

Figura 2: *Example of a Graph Cartesian Product*

Goal: we wish to characterize $rn(G \square H)$ in terms of $rn(G)$ and $rn(H)$ and possibly in terms of other properties of G and H.

Assume $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$.

2. Conditions Sufficient to Imply the Radio Condition

Let c be *any* labeling of $G \square H$, and assume c_G and c_H are radio labelings of G and H, respectively. We can show that c satisfies the radio condition on $(g_1,h_1),(g_2,h_2) \in V(G \square H)$ whenever
$$|c(g_1,h_1) - c(g_2,h_2)| \geq |c_G(g_1) - c_G(g_2)| + |c_H(h_1) - c_H(h_2)| - 1,$$
assuming $g_1 \neq g_2$ and $h_1 \neq h_2$. Similar conditions hold should $g_1 = g_2$ or $h_1 = h_2$.

3. The Diagonal Labeling Scheme

Index the rows of an $n \times m$ grid using $g_1, g_2, ..., g_n$ and the columns using $h_1, h_2, ..h_m$. The $(i,j)^{th}$ square represents the vertex (g_i, h_j).

The **Diagonal Labeling Scheme** lists the vertices of $G \square H$ in order of increasing label values. Figure 3 indicates how we label vertex (g_1,h_1) first, then move horizontally to vertex (g_1,h_2), descend a diagonal to vertex (g_2,h_1), move vertically to vertex (g_3,h_1), ascend a diagonal, etc.

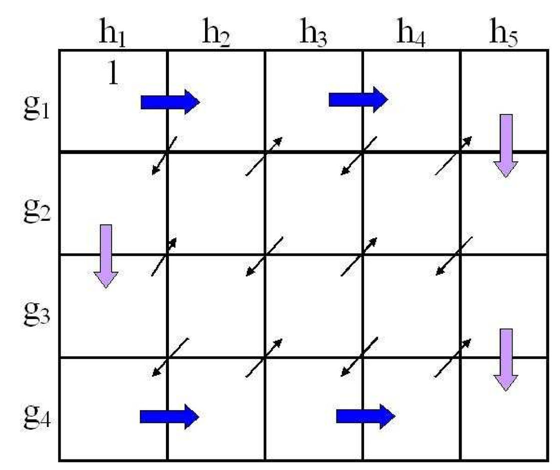

Figura 3: *Example of Diagonal Labeling Scheme*

4. An Upper Bound for $rn(G \square G)$

Teorema: Suppose $diam(G) = 2$ and G has a radio labeling c that uses labels with consecutive values (i.e. $rn(G) = |V(G)|$). Then
$$rn(G \square G) \leq n^2 + 2(2n-2) + 2\left(\sum_{i=1}^{n} \left\lfloor \frac{i-1}{2} \right\rfloor\right) - \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof Sketch:

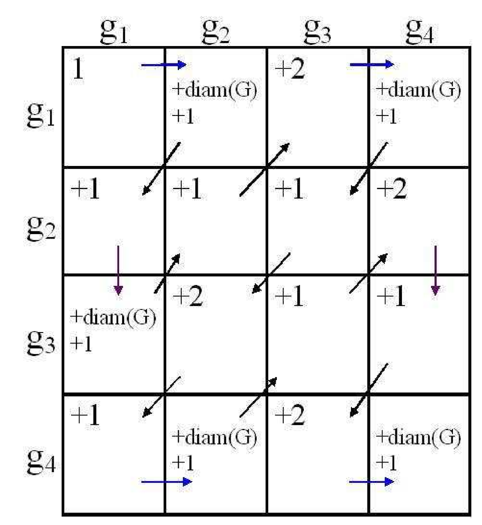

Figura 4: *Radio labeling $G \square G$*

The conditions referenced in Section 2 allow us to prove that this creates a radio labeling. We then calculate the span of the labeling:

- $(2n-2)$ is the number of times we move from one diagonal to another.
- $\left\lfloor \frac{i-1}{2} \right\rfloor$ is the number of "skips" within a diagonal of length i.
- $\sum_{i=1}^{n} \left\lfloor \frac{i-1}{2} \right\rfloor$ is the total number of skips within diagonals above and including the main diagonal.

5. An Upper Bound for $rn(G \square H)$

Teorema: Suppose the vertex sets of G and H may each be labeled using consecutive values, and that $\text{diam}(G) - \text{diam}(H) \geq 2$. Then
$$rn(G \square H) \leq \text{diam}(G)(n+m-2) + 2mn - 2m - 4n + 6 + k,$$
where $k = 0, 1,$ or 2, depending on the parities of $|V(G)|$ and $|V(H)|$.

Proof Sketch:

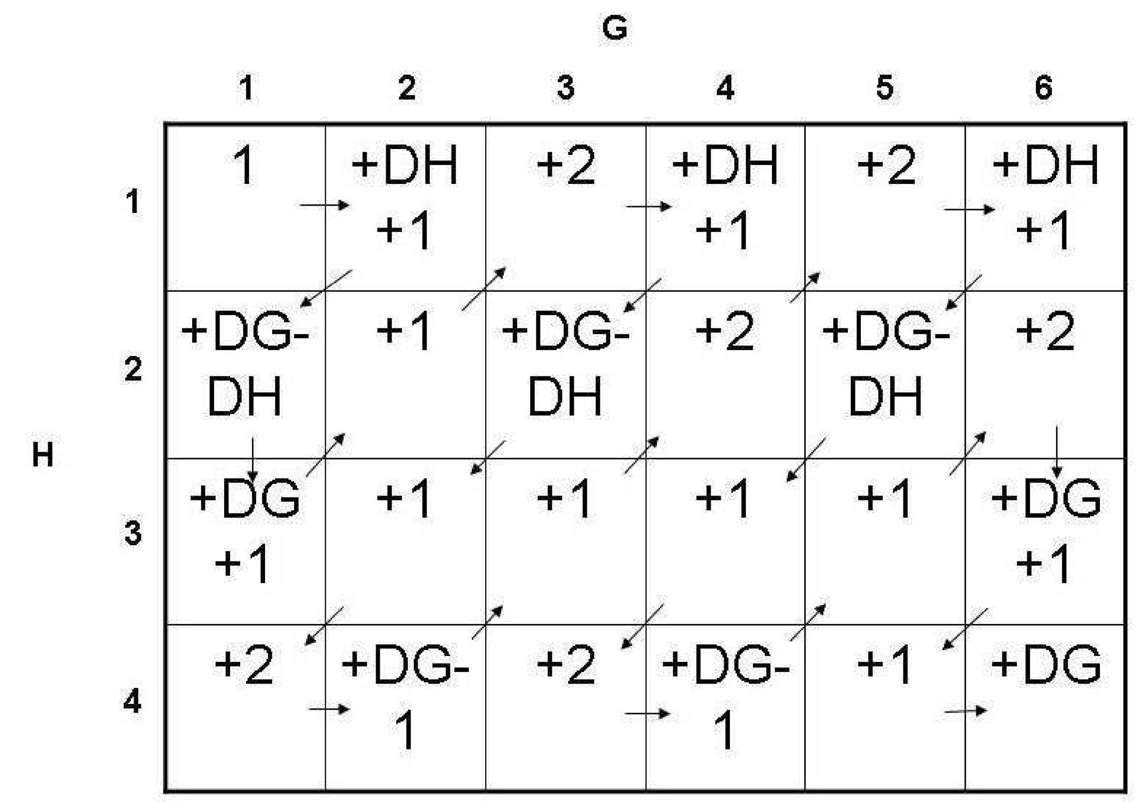

Figura 5: *Radio labeling $G \square H$*

Again the conditions referenced in Section 2 ensure we have a radio labeling, and the bound is given by calculating the span of this labeling.

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References

[1] Chartrand, Erwin, Harary, and Zhang, *Radio labeling of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77-85.