Radio Numbers of Kneser Graphs

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1. Context and Definitions

- The **distance** between any two vertices u, v of a graph G, $d(u,v)$, is the length of the shortest path between u and v. The **diameter** of a graph G, $\text{diam}(G)$, is the maximum distance between a pair of vertices in G.
- A **radio labeling** is a function $c : V(G) \to Z_+$ that satisfies the following **radio condition** for all distinct vertices u and v:

$$d(u,v) + |c(u) - c(v)| \geq \text{diam}(G) + 1.$$

The **span of** c, $span(c)$, is the maximum value of c.

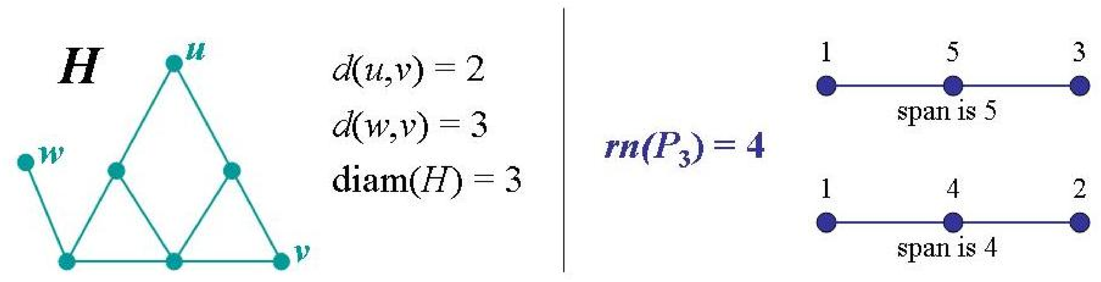

Figura 1: *Key examples of definitions*

- The **radio number of a graph** G, $rn(G)$, is the minimum span taken over all radio labelings of G.
- The **Kneser Graph** K_k^n is the graph whose vertices represent the k-subsets of $\{1, ..., n\}$ and where two vertices are adjacent if and only if they correspond to disjoint subsets.
- K_k^n has $\binom{n}{k}$ vertices and $\frac{\binom{n}{k}\binom{n-k}{k}}{2}$ edges.

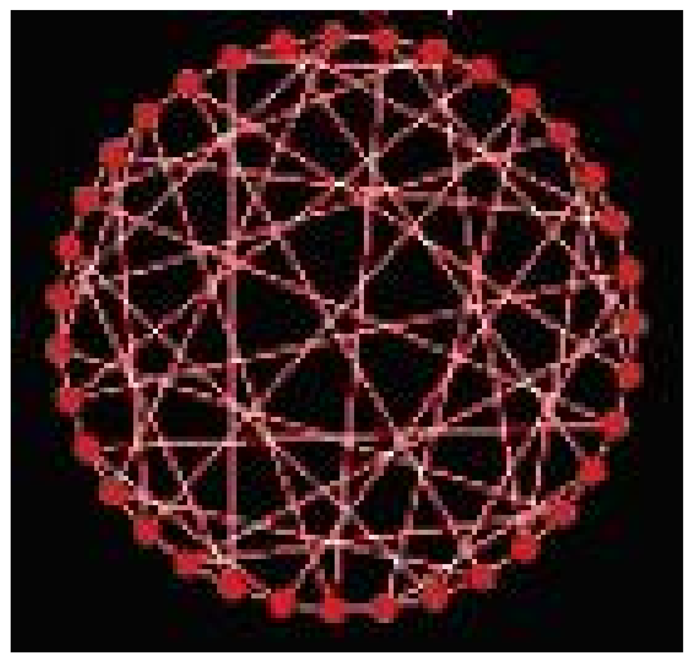

Figura 2: K_3^7 *(Taken from Mathworld)*

2. Kneser graphs of diameter 2

Goal: Prove that the radio number of K_k^n is $\binom{n}{k}$ if $\text{diam}(K_k^n) = 2$.

A Kneser graph K_k^n has a diameter of 2 if and only if $3k \leq n + 1$.

$$\text{If } u, v \text{ are vertices of } K_k^n\text{, then } d(u,v) = \begin{cases} 2, & \text{if } |u \cap v| \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 2.1 *If* $\text{diam}(K_k^n) = 2$*, then* $rn(K_k^n) = \binom{n}{k}$.

Proof:

Step 1: $rn(G) \geq |V(G)|$ **for any graph G.**

Proof of Step 1: Since $|c(u) - c(v)| \geq \text{diam}(G) - d(u,v) + 1 \geq 1$, no two vertices may have the same label value.

Step 2: $rn(K_k^n) \leq |V(K_k^n)|$ **when** $\text{diam}(K_k^n) = 2$.

Proof of Step 2: By algorithm. First we must define what a **block** is since we will be using them in our construction.

Definition 2.2 *The block* $B_{a,b}$ *is the lexicographically ordered list containing all ordered* k*-tuples with* (a,b) *as its first two elements.*

Example 2.3 *Let* $n = 7$ *and* $k = 4$. *Then* $B_{1,2} = ((1,2,3,4), (1,2,3,5), (1,2,3,6), (1,2,3,7), (1,2,4,5), (1,2,4,6), (1,2,4,7), (1,2,5,6), (1,2,5,7), (1,2,6,7))$.

Algorithm

- Let $B'_{a,b}$ be a list consisting of the elements of $B_{a,b}$ in reverse lexicographic order. Construct list of blocks such that every block except for $B_{1,2}$ is in reverse order (e.g. $B_{1,2}, B'_{1,3}, ..., B'_{1,n-k+2}, B'_{2,3}, ..., B'_{2,n-k+2}, ..., B'_{n-k+1,n-k+2}$).
- Then label the k-tuples in the blocks using consecutive integers.

This labeling satisfies the radio condition because consecutively labeled vertices correspond to non-disjoint subsets.

$$\textbf{Conclusion: } rn(K_k^n) = |V(K_k^n)| = \binom{n}{k}.$$

3. Further exploration (Kneser graphs of diameter 3 and above)

We know that the radio number of K_3^7 is 35, which is the number of vertices. However, it is unknown whether the radio number is equal to the number of vertices for Kneser graphs in general.

Following the suggestion of Dr. Cynthia Wyels, I used "diameter matrices". Diameter matrices are similar to adjacency matrices in that they express the existence of paths between vertices. However, diameter matrices display only the existence of paths of diameter length.

In a diameter matrix, if vertices v_i and v_j are a diameter apart, then the (i,j)th entry is 1. Otherwise the entry is zero.

Taking the permanent $\sum_{\pi \in S_n} M_{1\pi_1} M_{2\pi_2} \cdots M_{n\pi_n}$ of the diameter matrix tells us if there exists a sequence of vertices with diameter distances between every two consecutive vertices. If there is such a sequence, the permanent will be greater than 0, but if not, then the permanent is 0. If the permanent is 0, then $rn(K_k^n) > |V(K_k^n)|$.

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References

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