Let $U$ be a collection of subsets of the real numbers. In this collection $U$, consider two elements, $A$ and $B$ (they might be the same) and take the union of both, the intersection of both and the complement of each. Continue doing this with each pair of elements in $U$. What you are doing is building a new collection of subsets of the real numbers. Call this new family $U_1$. In this collection $U_1$, consider two elements, $A$ and $B$ (they might be the same) and repeat the same type of construction you did before: you are building a new collection of sets, call this new collection $U_2$. By iterating the process, you can construct $U_1, U_2, U_3, \ldots$. This raises several questions: If $U$ contains $n$ sets, how many elements does $U_1$ have? How about $U_2$? Does this sequence of collections “converge” in some sense, if so, to what does it converge? What are necessary and sufficient conditions for the sequence of collections to converge to some collection $V$? What properties does $V$ have? What if we start with $V$ now, what would $V_1$ be? What if the original collection $U$ contains infinitely many sets? Hypotheses about the nature of $U$ may be varied to address these questions through many perspectives.

These questions may be addressed by students initially familiar only with basic set definitions and operations, yet the questions become more challenging and interesting as they become more general. Analogies between these questions and ideas within the study of $\sigma$-algebras may be drawn: in the study of probability spaces and the Borel $\sigma$-algebra, one encounters a beautiful theorem that explains how the Borel $\sigma$-algebra can be constructed via similar operations. The theorem characterizes the elements of this $\sigma$-algebra in terms of intervals and countable sets (see [2]).

Investigations along these lines are appropriate for students who have completed coursework in Calculus, Discrete Mathematics or Logic, and a proofs-based course, ideally an upper-division Probability course. Students may form and test conjectures by constructing chains of families of sets; by comparing constructions the entire group can progress more rapidly. Background references and sources for ideas include the first chapters of [4], Chapter 5 of [1] and parts of [3] as well as a series of lectures on probability measures created by the advisor for this purpose.

This is a new area of investigation with applications to computer science, probability, and set theory. Any results of this type obtained by students would be new contributions and therefore publishable.

**References**