Radio numbers for generalized prism graphs

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Abstract

A radio labeling is an assignment $c : V(G) \rightarrow \mathbb{N}$ such that every distinct pair of vertices $u, v$ satisfies the inequality $d(u, v) + |c(u) - c(v)| \geq \text{diam}(G) + 1$. The span of a radio labeling is the maximum value of $c$. The radio number of $G$, $rn(G)$, is the minimum span over all radio labelings of $G$. Generalized prism graphs, denoted $Z_{n,s}$, $s \geq 1$, $n \geq s$, have vertex set $\{(i,j) | i = 1, 2 \text{ and } j = 1, \ldots, n\}$ and edge set $\{(i,j),(i,j \pm 1)\} \cup \{((1,i),(2,i + \sigma)) | \sigma = -\lfloor \frac{s-1}{2} \rfloor, \ldots, 0, \ldots, \lfloor \frac{s}{2} \rfloor\}$. In this paper we determine the radio number of $Z_{n,s}$ for $s = 1, 2$ and $3$. In the process we share techniques that are likely to be of use in determining radio numbers of other families of graphs.

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Radio labeling is a graph labeling problem, suggested by Chartrand, et al [1], that is analogous to assigning frequencies to FM channel stations so as to avoid signal interference. Radio stations that are close geographically must have frequencies that are very different, while radio stations with large geographical separation may have similar frequencies.

All graphs we consider are simple and connected. We denote by $V(G)$ the vertices of $G$. We use $d_G(u, v)$ for the length of the shortest path in $G$ between $u$ and $v$. The diameter of $G$, $diam(G)$, is the maximum distance in $G$. A radio labeling of $G$ is a function $c_G$ that assigns to each vertex $u \in V(G)$ a positive integer $c_G(u)$ such that any two distinct vertices $u$ and $v$ of $G$ satisfy the radio condition:

$$d_G(u, v) + |c_G(u) - c_G(v)| \geq diam(G) + 1.$$ 

The span of a radio labeling is the maximum value of $c_G$. Whenever $G$ is clear from context, we simply write $c(u)$ and $d(u, v)$. The radio number of $G$, $rn(G)$, is the minimum span over all possible radio labelings of $G$.

In this paper we determine the radio number of a family of graphs that consist of two $n$-cycle graphs together with some edges connecting vertices from different cycles. The motivating example for this family of graphs are the prism graphs, $Z_n,1$, which are the Cartesian product of $P_2$, the path on 2 vertices, and $C_n$, the cycle on $n$ vertices. In other words, prism graphs consist of two $n$-cycles with vertices labeled $(1,i)$, $i = 1, \ldots, n$ and $(2,i)$, $i = 1, \ldots, n$ respectively together with all edges between pairs of vertices of the form $(1,i)$ and $(2,i)$.

We will use pair notation to identify the vertices of the graphs with the first coordinate identifying the cycle, 1 or 2, and the second coordinate identifying the position of the vertex within the cycle, 1, ..., $n$. To avoid complicated notation, identifying a vertex as $(i, j)$ will always imply that the first coordinate is taken modulo 2 with $i \in \{1, 2\}$ and the second coordinate is taken modulo $n$ with $j \in \{1, \ldots, n\}$.

**Definition 1** Generalized prism graphs, denoted $Z_{n,s}$, $s \geq 1$, $n \geq s$, have vertex set $\{(i,j) \mid i = 1, 2 \text{ and } j = 1, \ldots, n\}$. Vertex $(i,j)$ is adjacent to $(i, j \pm 1)$. In addition, $(1,i)$ is adjacent to $(2, i + \sigma)$ for each $\sigma$ in $\{-\left\lfloor \frac{s+1}{2} \right\rfloor, \ldots, 0, \ldots, \left\lceil \frac{s}{2} \right\rceil\}$.

The two $n$-cycle subgraphs of $Z_{n,s}$ induced by 1) all vertices of the form $(1,j)$ and 2) all vertices of the form $(2,j)$ are called principal cycles.

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Footnote 1: We use the convention that $\mathbb{N}$ consists of the positive integers. Some authors let $\mathbb{N}$ include 0, with the result that radio numbers using this definition are one less than radio numbers determined using the positive integers.
Fig. 1. Z_{8,1}, Z_{8,2}, and Z_{8,3}, with standard cycles depicted by dashed edges

In this notation, the prism graphs $C_n \square P_2$ are $Z_{n,1}$. We note that $Z_{n,2}$ graphs are isomorphic to the squares of even cycle graphs, $C_{2n}^2$ whose radio number is determined in [3]. The graphs $Z_{8,1}, Z_{8,2},$ and $Z_{8,3}$ are illustrated in Figure 1.

Remark 2 We note that

- $\text{diam}(Z_{n,1}) = \text{diam}(C_n) + \text{diam}(P_2) = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{n+2}{2} \right\rfloor$
- $\text{diam}(Z_{n,2}) = \left\lfloor \frac{n+1}{2} \right\rfloor$
- $\text{diam}(Z_{n,3}) = \left\lfloor \frac{n}{2} \right\rfloor$.

In general, $\text{diam}(Z_{n,s}) = \left\lfloor \frac{n+3-s}{2} \right\rfloor$ for $s = 1, 2, 3$.

Our general approach to determining the radio number of $Z_{n,s}$ consists of two steps. We first establish a lower bound for the radio number. Suppose $c$ is a radio labeling of the vertices of $Z_{n,s}$. We can rename the vertices of $Z_{n,s}$ with $\{\alpha_i \mid i = 1, \ldots, 2n\}$, so that $c(\alpha_i) < c(\alpha_j)$ whenever $i < j$. We determine the minimum label difference between $c(\alpha_i)$ and $c(\alpha_{i+2})$, denoted $\phi(n, s)$, and use it to establish that $rn(Z_{n,s}) \geq 2 + (n - 1)\phi(n, s)$. We then demonstrate an algorithm that establishes that this lower bound is in fact the radio number of the graph. We do this by defining a position function $p : V(G) \to \{\alpha_i \mid i = 1, \ldots, 2n\}$ and a labeling function $c : \{\alpha_i\} \to \mathbb{Z}^+$ that has span $(n-1)\phi(n, s)+2$.

We prove that $p$ is a bijection, i.e., every vertex is labeled exactly once, and that all pairs of vertices together with $c \circ p^{-1}$ satisfy the radio condition.

Some small cases of generalized prism graphs with $s = 3$ do not follow the general pattern, so we discuss these first. First note that $Z_{3,3}$ has diameter 1 and thus can be radio-labeled using consecutive integers, i.e., $rn(Z_{3,3}) = 6$.

To determine $rn(Z_{4,3})$, note that in a radio labeling two vertices can have consecutive labels only when the distance between them is equal to the diameter of the graph. The diameter of $Z_{4,3}$ is 2. If two vertices of $Z_{4,3}$ are at distance 2, their second coordinates have the same parity. Consider examining the vertices of $Z_{4,3}$ in increasing order of the values of their labels in some radio labeling. At some point the list must – due to changing the parity of the
second coordinate – proceed from a vertex \( v \) to a vertex adjacent to \( v \). Thus it is impossible to label \( Z_{4,3} \) using consecutive integers and so \( rn(Z_{4,3}) \geq 9 \). We leave it to the reader to verify that \( rn(Z_{4,3}) = 9 \).

To simplify many of the computations that follow, we make use of the existence of certain special cycles in the graphs.

**Definition 3** Suppose a graph \( G \) contains a subgraph \( H \) isomorphic to a cycle graph, and let \( v \in V(H) \).

- We will call \( H \) a \( v \)-tight cycle if for every \( u \in V(H) \), \( d_G(u, v) = d_H(u, v) \).
- We will call \( H \) a tight cycle if for every pair of vertices \( u, w \) in \( H \), \( d_G(u, w) = d_H(u, w) \).

We note that \( H \) is a tight cycle if and only if \( H \) is \( v \)-tight for every \( v \in V(H) \).

**Remark 4** Each of the two principal \( n \)-cycles is tight.

Particular tight cycles of maximum length play an important role in our proofs. Figure 1 uses dashed edges to indicate a tight cycle of maximum length for each of the three types of graphs under consideration. We will call these particular maximum-length tight cycles standard. For convenience, we will use a second set of names for the vertices of a standard cycle when focusing on properties of, or distance within, the standard cycles. The vertices of a standard cycle for \( Z_{n,s} \) will be labeled \( X_i^s \), \( i = 1, \ldots, n + 3 - s \), where

\[
X_i^1 = \begin{cases} 
(1,1), & \text{if } i = 1, \\
(1,2), & \text{if } i = 2, \\
(2, i - 1) & \text{if } i \geq 3.
\end{cases}
\]

and for \( k = 2, 3 \),

\[
X_i^k = \begin{cases} 
(1,1), & \text{if } i = 1, \\
(2,i) & \text{if } i \geq 2.
\end{cases}
\]

These labels are illustrated in Figure 1.

**Remark 5** The standard cycles depicted in Figure 1 are \((1,1)\)-tight and have \( n + 3 - s \) vertices. Therefore each standard cycle has diameter equal to the diameter of its corresponding \( Z_{n,s} \) graph.
1 Lower Bound

Suppose $c$ is a radio labeling of $G$ with minimal span. Intuitively, building such a labeling requires one to find groups of vertices that are pairwise far from each other so they can be assigned labels that have small pairwise differences. The following preliminary lemma will be used to determine the maximal pairwise distance in a group of 3 vertices in $Z_{n,s}$. This leads to Lemma 8, in which we determine the minimal difference between the largest and smallest label in any group of 3 vertex labels.

**Lemma 6** Let \{\(u, v, w\)\} be any subset of \(V(Z_{n,s})\), \(s \leq 3\), with the exception of \{(1, j), (2, j), (i, l)\} in \(V(Z_{n,3})\). Then \(d(u, v) + d(v, w) + d(u, w) \leq n + 3 - s\).

**PROOF.**

**Claim 7** If \(u, v, \) and \(w\) lie on a cycle of length \(t\), then \(d(u, w) + d(v, u) + d(w, u) \leq t\).

The three vertices separate the cycle into three disjoint paths. The length of each path gives an upper bound for the distance between its endpoints. As the sum of the lengths of these paths is equal to \(t\), \(d(u, w) + d(v, u) + d(w, u) \leq t\), which proves the claim.

If \(u, v,\) and \(w\) lie on the same principal \(n\)-cycle, the desired result follows directly from the claim, as all three vertices lie on a cycle of length \(n\), and \(s \leq 3\).

Suppose \(u, v, \) and \(w\) do not all lie on the same principal \(n\)-cycle. Without loss of generality, assume \(u = (1, 1)\), and \(v\) and \(w\) lie on the second principal \(n\)-cycle. Then for \(s = 1\) or \(2\), the standard cycle includes \((1, 1)\) and all vertices \((2, i)\), so \(v\) and \(w\) lie on the standard cycle. For \(s = 3\), the standard cycle includes all vertices \((2, i)\), \(i > 1\). As the triple \{(1, j), (2, j), (i, l)\} in \(V(Z_{n,3})\) was eliminated in the hypothesis, it follows that for \(s \leq 3\) all three vertices lie on the appropriate standard cycle. As the standard cycle in each case is of length \(n - s + 3\), the result follows by the claim. \(\square\)

**Lemma 8** Let \(c(Z_{n,s})\) be a radio labeling of \(Z_{n,s}\), \(s \leq 3\) and \(n = 4k + r\), where \(k \geq 1\), \(r = 0, 1, 2, 3\), and \((n, s) \neq (4, 3)\). Suppose \(V(G) = \{\alpha_i | i = 1, ..., 2n\}\) and \(c(\alpha_i) < c(\alpha_j)\) whenever \(i < j\). Then we have \(|c(\alpha_{i+2}) - c(\alpha_i)| \geq \phi(n, s)\),
where the values of $\phi(n, s)$ are given in the following table.

<table>
<thead>
<tr>
<th>$\phi(n, s)$</th>
<th>$s=1$</th>
<th>$s=2$</th>
<th>$s=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r=0$</td>
<td>$k+2$</td>
<td>$k+1$</td>
<td>$k+2$</td>
</tr>
<tr>
<td>$r=1$</td>
<td>$k+2$</td>
<td>$k+2$</td>
<td>$k+1$</td>
</tr>
<tr>
<td>$r=2$</td>
<td>$k+3$</td>
<td>$k+2$</td>
<td>$k+2$</td>
</tr>
<tr>
<td>$r=3$</td>
<td>$k+2$</td>
<td>$k+3$</td>
<td>$k+2$</td>
</tr>
</tbody>
</table>

**PROOF.**

First assume $\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}$ are any three vertices in any generalized prism graph with $s \leq 3$ except $\{(1, j), (2, j), (i, l)\}$ in $V(Z_{n,3})$. Apply the radio condition to each pair in the vertex set $\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}$ and take the sum of three inequalities. We obtain

$$d(\alpha_{i+1}, \alpha_i) + d(\alpha_{i+2}, \alpha_{i+1}) + d(\alpha_{i+2}, \alpha_i) + |c(\alpha_{i+1}) - c(\alpha_i)| + |c(\alpha_{i+2}) - c(\alpha_{i+1})| + |c(\alpha_{i+2}) - c(\alpha_i)| \geq 3 \text{diam}(Z_{n,s}) + 3. \quad (1)$$

We drop the absolute value signs because $c(\alpha_i) < c(\alpha_{i+1}) < c(\alpha_{i+2})$, and use Lemma 6 to rewrite the inequality as

$$c(\alpha_{i+2}) - c(\alpha_i) \geq \frac{1}{2} \left(3 + 3 \text{diam}(Z_{n,s}) - (n + 3 - s)\right).$$

The table in the statement of the Lemma has been generated by substituting the appropriate values for $\text{diam}(Z_{n,s})$ from Remark 2 and simplifying. As the computations are straightforward but tedious, they are not included.

It remains to consider the case $\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\} = \{(1, j), (2, j), (i, l)\}$ in $V(Z_{n,3})$. From the radio condition, it follows that

$$d((1, j), (2, j)) + |c(1, j) - c(2, j)| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

and so

$$|c(1, j) - c(2, j)| \geq \left\lfloor \frac{n}{2} \right\rfloor \geq 2k.$$
Thus we may conclude
\[ |c(\alpha_{i+2}) - c(\alpha_i)| \geq |c(1, j) - c(2, j)| \geq 2k. \]

If \( k \geq 2 \), then
\[ |c(\alpha_{i+2}) - c(\alpha_i)| \geq 2k \geq k + 2 \geq \phi(n, 3). \]

If \( k = 1 \), recall that \( Z_{4,3} \) is excluded in the hypothesis. It is easy to verify that for \( n = 5, 6, 7 \), \( \phi(n, 3) = \left\lfloor \frac{n}{2} \right\rfloor \). \( \square \)

**Remark 9** For all values of \( n \) and \( s \leq 3 \), \( 2\phi(n, s) \geq \text{diam}(Z_{n,s}) \).

**Theorem 10** For every graph \( Z_{n,s} \) with \( s \leq 3 \),
\[ rn(Z_{n,s}) \geq (n - 1)\phi(n, s) + 2. \]

**PROOF.** We may assume \( c(\alpha_1) = 1 \). By Lemma 8, \( |c(\alpha_{i+2}) - c(\alpha_i)| \geq \phi(n, s) \), so \( c(\alpha_{2i-1}) = c(\alpha_{1+2(i-1)}) \geq (i - 1)\phi(n, s) + 1 \). Note that all generalized prism graphs have \( 2n \) vertices. As
\[ c(\alpha_{2n-1}) \geq (n - 1)\phi(n, s) + 1, \]
we have
\[ c(\alpha_{2n}) \geq c(\alpha_{2n-1}) + 1 = (n - 1)\phi(n, s) + 2. \]
\( \square \)

2 Upper Bound

To construct a labeling for \( Z_{n,s} \) we will define a position function \( p : \{\alpha_i \mid i = 1, \ldots, 2n\} \rightarrow V(Z_{n,s}) \) and then a labeling function \( c : \{\alpha_i \mid i = 1, \ldots, 2n\} \rightarrow \mathbb{N} \). The composition \( c \circ p^{-1} \) gives an algorithm to label \( Z_{n,s} \), and this labeling has span equal to the lower bound found in Theorem 10. The labeling function depends only on the function \( \phi(n, s) \) defined in Lemma 8.

**Definition 11** Let \( \{\alpha_i \mid i = 1, \ldots, 2n\} \) be the vertices of \( Z_{n,s} \). Define \( c : \{\alpha_i \mid i = 1, \ldots, 2n\} \rightarrow \mathbb{N} \) to be the function
\[ c(\alpha_{2i-1}) = 1 + (i - 1)\phi(n, s), \]
and
\[ c(\alpha_{2i}) = 2 + (i - 1)\phi(n, s). \]

Suppose \( f \) is any labeling of any graph \( G \). The “diameter condition”
\[
|f(u) - f(v)| \geq \text{diam}(G) \quad \text{for some} \quad u, v \in V(G)
\]
is sufficient to satisfy the radio condition as applied to \( f, u, \) and \( v \). The next lemma identifies pairs of vertices of \( Z_{n,s} \) for which the labeling \( c \) of Definition 11 satisfies the diameter condition. It will be used to limit the number of vertex pairs for which it must be checked that specific labelings satisfy the radio condition.

**Lemma 12** Let \( \{\alpha_i | i = 1, ..., 2n\} \) be the vertices of \( Z_{n,s} \) and \( c \) be the labeling of Definition 11. Then whenever \( |l - k| \geq 4 \),
\[
d(\alpha_l, \alpha_k) + |c(\alpha_l) - c(\alpha_k)| \geq \text{diam}(Z_{n,s}) + 1.
\]

**PROOF.** Without loss of generality, let \( l > k \). Since \( c(\alpha_{k+4}) \leq c(\alpha_l) \), it follows that
\[
c(\alpha_l) - c(\alpha_k) \geq c(\alpha_{k+4}) - c(\alpha_k) = 2\phi(n, s).
\]

From Remark 9 it follows that
\[
|c(\alpha_l) - c(\alpha_k)| + d(\alpha_l, \alpha_k) \geq 2\phi(n, s) + 1 \geq \text{diam}(Z_{n,s}) + 1. \quad \square
\]

We will need to consider four different position functions depending on \( n \) and \( s \).

**Algorithm 1:** \( n = 4k + r \), \( r = 1, 2, 3 \) and \( s \leq 3 \)

except \( n = 4k + 2 \) when \( k \) is even and \( s = 3 \)

The idea behind the algorithm is to find a position function which allows pairs of consecutive integers to be used as labels. To use consecutive integers we need to find pairs of vertices in \( Z_{n,s} \) with distance equal to the diameter. We will do that by taking advantage of the standard cycles for each value of \( s \).

**Remark 13** Name the vertices of a \( t \)-cycle, \( C_t \), using \( \{1, 2, ..., t\} \) where \( i \) is adjacent to \( i - 1 \) and to \( i + 1 \), both modulo \( t \). Then \( d \left( i, i + \left[\frac{t + 1}{2}\right]\right) = \text{diam}(C_t)\).
Lemma 14 For all \( n \geq 3 \) and \( s \leq 3 \), \( d((1,y),(2,y+D)) = \text{diam}(Z_{n,s}) \) when
\[
D = \begin{cases} 
\lfloor \frac{n+1}{2} \rfloor, & \text{for } s = 1 \text{ and } 3, \\
\lfloor \frac{n+2}{2} \rfloor, & \text{for } s = 2.
\end{cases}
\]

PROOF.

As all vertices lying on the same principal cycle of \( Z_{n,s} \) are equivalent, we may assume that \((1,y) = (1,1)\). Consider the standard cycle in \( Z_{n,s} \). Then \((1,1) = X_1^s\) and \((2,1+D) = X_{\lfloor \frac{n+3}{2} \rfloor +1}^{s+1} \). The result follows by the observation that the standard cycle in each case is isomorphic to \( C_{n+3-s} \) and by Remark 13.

The position function for Algorithm 1 is
\[
p_1(\alpha_{2i-1}) = (1,1+\omega(i-1)) \quad \text{and} \\
p_1(\alpha_{2i}) = (2,1+D+\omega(i-1)),
\]
where \( D \) is as defined in Lemma 14 and
\[
\omega = \begin{cases} 
k, & \text{if } n = 4k + 2 \text{ when } k \text{ is odd, and } n = 4k + 1, \\
k+1, & \text{if } n = 4k + 2 \text{ when } k \text{ is even, and } n = 4k + 3.
\end{cases}
\]

Lemma 15 The function \( p_1 : \{\alpha_j \mid j = 1, ..., 2n\} \to V(Z_{n,s}) \) is a bijection.

PROOF.

Suppose \( p_1(\alpha_a) = p_1(\alpha_b) \) with \( a > b \). Let \( i = \lfloor \frac{a}{2} \rfloor \) and \( j = \lfloor \frac{b}{2} \rfloor \). As \( p_1(\alpha_a) \) and \( p_1(\alpha_b) \) have the same first coordinate, \( a \) and \( b \) have the same parity. Examining the second coordinates we can conclude that \( i\omega \equiv j\omega \mod n \) or \((i-j)\omega \equiv 0 \mod n \).

By Euclid’s algorithm, \( k \) is co-prime to \( 4k+1 \) and \( k \) is co-prime to \( 4k+2 \) when \( k \) is odd. Also, \( k+1 \) is co-prime to \( 4k+2 \) when \( k \) is even and \( k+1 \) is co-prime to \( 4k+3 \) for all \( k \). Thus in all cases \( \gcd(n,\omega) = 1 \). As \( n \) divides \((i-j)\omega \), it follows that \( n \) divides \((i-j)\). But then \((i-j) \geq n \) and thus \( a-b \geq 2n \), so \( a > 2n \), a contradiction.

Lemma 15 establishes that the function \( c \circ p^{-1} : V(Z_{n,s}) \to \mathbb{N} \) assigns each vertex exactly one label. It remains to show that the labeling satisfies the radio
condition. The following remark simplifies many of the calculations needed.

**Remark 16** We verify in the Appendix that in all cases considered,

- \( \phi(n, s) + \omega \geq \text{diam}(Z_{n,s}) + 1 \) and that
- \( \phi(n, s) - \omega \geq \begin{cases} 1, & \text{if } n - s \text{ is even} \\ 2, & \text{if } n - s \text{ is odd.} \end{cases} \)

**Theorem 17** The function \( c \circ p_1^{-1} : V(Z_{n,s}) \to \mathbb{N} \) defines a radio labeling on \( Z_{n,s} \) for the values of \( n \) and \( s \) considered by this algorithm.

**PROOF.** By Lemma 6 it is enough to check that all pairs of vertices in the set \( \alpha_j, .., \alpha_{j+3} \) satisfy the radio condition. As \( d(\alpha_j, \alpha_{j+a}) \) depends only on \( a \) and on the parity of \( j \), it is enough to check all pairs of the form \( (\alpha_{2i-1}, \alpha_{2i-1+a}) \) and all pairs of the form \( (\alpha_{2i}, \alpha_{2i+a}) \) for \( a \leq 3 \) and for some \( i \). To simplify the computations, we will check these pairs in the case when \( i = 1 \). For the convenience of the reader, we give the coordinates and the labels of the relevant vertices.

<table>
<thead>
<tr>
<th>vertex ( \alpha_j )</th>
<th>label value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1,1) )</td>
<td>1</td>
</tr>
<tr>
<td>( (2,1 + D) )</td>
<td>2</td>
</tr>
<tr>
<td>( (1,1 + \omega) )</td>
<td>1 + ( \phi(n,s) )</td>
</tr>
<tr>
<td>( (2,1 + D + \omega) )</td>
<td>2 + ( \phi(n,s) )</td>
</tr>
<tr>
<td>( (1,1 + 2\omega) )</td>
<td>1 + 2( \phi(n,s) )</td>
</tr>
</tbody>
</table>

**Pair** \( (\alpha_1, \alpha_2) \): By Lemma 14, \( d(\alpha_1, \alpha_2) = \text{diam}(Z_{n,s}) \). Thus \( d(\alpha_1, \alpha_2) + c(\alpha_2) - c(\alpha_1) = \text{diam}(Z_{n,s}) + 1 \), as required.

**Pairs** \( (\alpha_1, \alpha_3) \) and \( (\alpha_2, \alpha_4) \): Note that \( \alpha_1 \) and \( \alpha_3 \) lie on the same principal \( n \)-cycle and this cycle is tight by Remark 4. Thus

\[
d(\alpha_1, \alpha_3) + c(\alpha_3) - c(\alpha_1) = \omega + \phi(n,s) \geq \text{diam}(Z_{n,s}) + 1.
\]

The last inequality follows by Remark 16. The relationship between \( \alpha_2 \) and \( \alpha_4 \) is identical.

**Pair** \( (\alpha_1, \alpha_4) \): Note that

\[
d(\alpha_1, \alpha_4) \geq d(\alpha_1, \alpha_2) - d(\alpha_2, \alpha_4) = \text{diam}(Z_{n,s}) - \omega.
\]
Thus
\[ d(\alpha_1, \alpha_4) + c(\alpha_4) - c(\alpha_1) \geq \text{diam}(Z_{n,s}) - \omega + \phi(n, s) + 1 \geq \text{diam}(Z_{n,s}) + 2, \]
where the last inequality follows by Remark 16.

**Pair** \((\alpha_2, \alpha_3):\) By subtracting \(\omega\) from the second coordinate, we see \(d(\alpha_2, \alpha_3) = d(((1, 1), (2, 1 + D - \omega)).\)

When considered in the standard cycle, these vertices correspond to \(X_1^1\) and \(X_2^1+D-\omega\) if \(s = 1\) and to \(X_1^k\) and \(X_1^1+D-\omega\) if \(s = 2\) or 3. As by Remark 5 the standard cycle in each case is \(X_1^s\)-tight, we have
\[
d(\alpha_2, \alpha_3) = \begin{cases} D - \omega + 1 = \left\lceil \frac{n+3}{2} \right\rceil - \omega, & s = 1, \\ D - \omega = \left\lceil \frac{n+2}{2} \right\rceil - \omega, & s = 2, \\ D - \omega = \left\lceil \frac{n+1}{2} \right\rceil - \omega, & s = 3. \end{cases}
\]

Thus in all cases \(d(\alpha_2, \alpha_3) = \left\lceil \frac{n+3-s+1}{2} \right\rceil - \omega,\) so
\[
d(\alpha_2, \alpha_3) \geq \left\lceil \frac{n+3-s+1}{2} \right\rceil - \omega \tag{3}
\]
\[
= \begin{cases} \text{diam}(Z_{n,s}) + 1 - \omega, & n - s \text{ even}, \\ \text{diam}(Z_{n,s}) - \omega, & n - s \text{ odd}. \end{cases}
\]

By Remark 16, \(\phi(n, s) - \omega \geq 1\) when \(n - s\) is even and \(\phi(n, s) - \omega \geq 2\) when \(n - s\) is odd. Thus
\[
d(\alpha_2, \alpha_3) + c(\alpha_3) - c(\alpha_2) \geq \left\lceil \frac{n+3-s+1}{2} \right\rceil - \omega + (\phi(n, s) - 1)
\]
\[
\geq \begin{cases} \text{diam}(Z_{n,s}) + 1 + 1 - 1, & n - s \text{ even}, \\ \text{diam}(Z_{n,s}) + 2 - 1, & n - s \text{ odd}. \end{cases}
\]

**Pair** \((\alpha_2, \alpha_5):\) As \(|c(\alpha_5) - c(\alpha_2)| = 2\phi(n, s) - 1\) and \(d(\alpha_2, \alpha_5) \geq 1\), it follows that \(|c(\alpha_5) - c(\alpha_2)| + d(\alpha_2, \alpha_5) \geq 2\phi(n, s),\) and so by Remark 9, \(|c(\alpha_5) - c(\alpha_2)| + d(\alpha_2, \alpha_5) \geq \text{diam}(Z_{n,s}) + 1.\)

This establishes that \(c \circ p_1^{-1}\) is a radio labeling of \(Z_{n,s}\). \(\Box\)
The position function is

\[ p_2(\alpha_{2i-1}) = (1 + l_i, 1 + k(i - 1) - l_i), \text{ and} \]
\[ p_2(\alpha_{2i}) = (2 + l_i, 1 + k(i + 1) - l_i), \]

\[(4)\]

where \( l_i = \left\lfloor \frac{i-1}{4} \right\rfloor. \]

To simplify notation, we will also denote the value of \( l \) associated to a particular vertex \( v \) by \( l(v) \). Note that \( l(\alpha_{2n}) = l_n = l_{4k} = \left\lfloor \frac{4k-1}{4} \right\rfloor \leq k - 1. \)

**Lemma 18** The function \( p_2 : \{\alpha_j \mid j = 1, \ldots, 2n\} \rightarrow V(Z_{n,s}) \) is a bijection.

**PROOF.** To show that \( p_2 \) is a bijection, suppose that \( p_2(\alpha_a) = p_2(\alpha_b) \) and let \( i = \left\lfloor \frac{a}{2} \right\rfloor \) and \( j = \left\lfloor \frac{b}{2} \right\rfloor. \) Suppose first that \( a \) and \( b \) are even. Then

\[ 1 + k(i - 1) - l(a) \equiv 1 + k(j - 1) - l(b) \mod n. \]

Thus

\[ k(i - j) + l(b) - l(a) \equiv 0 \mod 4k, \]

so \( l(b) - l(a) \equiv 0 \mod k. \) As \( l(b) - l(a) \leq k - 1, \) this implies that \( l(b) = l(a). \) Then we have that

\[ k(i - j) + l(b) - l(a) = k(i - j) \equiv 0 \mod 4k, \]

and thus \( i - j \geq 4 \) or \( a - b \geq 8. \) However, \( a - b \geq 8 \) implies that \( l(b) \neq l(a), \) a contradiction. The argument when \( a \) and \( b \) are odd is similar.

Suppose then that \( a \) is even and \( b \) is odd. Then \( 1 + l(b) \equiv 2 + l(a) \mod 2 \) shows that \( l(a) \) and \( l(b) \) have different parity and in particular \( l(a) - l(b) \neq 0. \) On the other hand, considering the second coordinates of \( p_2(\alpha_a) \) and \( p_2(\alpha_b) \mod k, \) we deduce that \( 1 - l(a) \equiv 1 - l(b) \mod k \) or \( l(a) - l(b) \equiv 0 \mod k. \) As \( l(a) - l(b) \neq 0 \) it follows that \( |l(a) - l(b)| \geq k, \) a contradiction. \( \square \)

**Lemma 19** The function \( c \circ p_2^{-1} : V(Z_{n,s}) \rightarrow N \) defines a radio labeling on \( Z_{4k,1} \) and \( Z_{4k,3}. \)

**PROOF.** The inequality the function must satisfy when applied to \( Z_{4k,1} \) is \( d(u, v) + |c(u) - c(v)| \geq \text{diam}(Z_{4k,1}) + 1 = 2k + 2. \) For \( Z_{4k,3}, \) the corresponding inequality is \( d(u, v) + |c(u) - c(v)| \geq \text{diam}(Z_{4k,3}) + 1 = 2k + 1. \)
For both $Z_{4k,1}$ and $Z_{4k,3}$, $\phi(n, s) = k + 2$, thus $c_{Z_{4k,1}}(u) = c_{Z_{4k,3}}(u)$. If $u$ and $v$ are in the same principal cycle, then $d_{Z_{n,3}}(u, v) = d_{Z_{n,1}}(u, v)$, as principal cycles are always tight. If $u$ and $v$ are on different principal cycles, it is easy to verify that $d_{Z_{n,3}}(u, v) = d_{Z_{n,1}}(u, v) - 1$ by comparing the standard $u$-tight cycles on the two graphs. Thus we can conclude that $d_{Z_{n,3}}(u, v) \geq d_{Z_{n,1}}(u, v) - 1$ and so if the radio condition is satisfied by $c_{Z_{n,1}}$, the corresponding radio condition is satisfied by $c_{Z_{n,3}}$. We will check the radio condition assuming that $s = 1$.

As before, it suffices to check that the radio condition holds for all pairs of the form $(\alpha_{2i-1}, \alpha_{2i-1+a})$ and all pairs of the form $(\alpha_{2i}, \alpha_{2i+a})$ for $a \leq 3$. For the convenience of the reader, the relevant values of $p_2$ and $c$ are provided below.

<table>
<thead>
<tr>
<th>vertex</th>
<th>label value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{2i-1}$</td>
<td>$(1 + l_i, 1 + k(i - 1) - l_i)$</td>
</tr>
<tr>
<td>$\alpha_{2i}$</td>
<td>$(2 + l_i, 1 + k(i + 1) - l_i)$</td>
</tr>
<tr>
<td>$\alpha_{2(i+1)-1}$</td>
<td>$(1 + l_{i+1}, 1 + ki - l_{i+1})$</td>
</tr>
<tr>
<td>$\alpha_{2(i+1)}$</td>
<td>$(2 + l_{i+1}, 1 + k(i + 2) - l_{i+1})$</td>
</tr>
<tr>
<td>$\alpha_{2(i+2)-1}$</td>
<td>$(1 + l_{i+2}, 1 + k(i + 1) - l_{i+2})$</td>
</tr>
</tbody>
</table>

Note that $l_{(\alpha_{r+a})} - l_{(\alpha_r)} = 0$ or 1 whenever $a \leq 3$. As $s = 1, d((x_1, y_1)(x_2, y_2)) = |x_2 - x_1| + \min\{|y_2 - y_1|, 4k - |y_2 - y_1|\}$. The following table has been generated using this equation.

<table>
<thead>
<tr>
<th>vertex pair</th>
<th>$d(u, v)$</th>
<th>$d(u, v)$</th>
<th>$d(u, v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2i})$</td>
<td>$</td>
<td>l_{(u)} - l_{(v)}</td>
<td>= 0$</td>
</tr>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2(i+1)-1})$</td>
<td>$0 + k$</td>
<td>$1 + (k - 1)$</td>
<td>$k + 2$</td>
</tr>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2(i+1)})$</td>
<td>$1 + k$</td>
<td>$0 + (k + 1)$</td>
<td>$k + 3$</td>
</tr>
<tr>
<td>$(\alpha_{2i}, \alpha_{2(i+1)-1})$</td>
<td>$1 + k$</td>
<td>$0 + (k + 1)$</td>
<td>$k + 1$</td>
</tr>
<tr>
<td>$(\alpha_{2i}, \alpha_{2(i+1)})$</td>
<td>$0 + k$</td>
<td>$1 + (k - 1)$</td>
<td>$k + 2$</td>
</tr>
<tr>
<td>$(\alpha_{2i}, \alpha_{2(i+2)-1})$</td>
<td>$1 + 0$</td>
<td>$0 + 1$</td>
<td>$2k + 3$</td>
</tr>
</tbody>
</table>

It is straightforward to verify that in each case, $d(u, v) + |c(u) - c(v)| \geq 2k + 2$. □

13
Algorithm 3: \( n = 4k, s = 2 \)

The position function for Algorithm 3 is

\[
p_3(\alpha_{2i-1}) = (i, 1 + k(i - 1) - l_i), \quad \text{and} \quad p_3(\alpha_{2i}) = (i, 1 + k(i + 1) - l_i),
\]

where \( l_i = \left\lfloor \frac{i - 1}{2} \right\rfloor \). Note that \( l_{2n} = l_{4k} = \left\lfloor \frac{4k - 1}{2} \right\rfloor \leq 2k - 1 \).

**Lemma 20** The function \( p_3 : \{ \alpha_j | j = 1, .., 2n \} \to V(Z_{n,s}) \) is a bijection.

**PROOF.**

Suppose that \( p_3(\alpha_a) = p_3(\alpha_b) \) and let \( i = \left\lfloor \frac{a}{2} \right\rfloor \) and \( j = \left\lfloor \frac{b}{2} \right\rfloor \). First suppose \( a \) and \( b \) have the same parity, say even. Then

\[
1 + k(i + 1) - l_{(a)} \equiv 1 + k(j + 1) - l_{(b)} \mod n.
\]

Thus

\[
k(i - j) + l_{(b)} - l_{(a)} \equiv 0 \mod 4k,
\]

so \( l_{(b)} - l_{(a)} \equiv 0 \mod k \). As \( |l_{(a)} - l_{(b)}| \leq 2k - 1 \), this implies that \( l_{(a)} = l_{(b)} \) or that \( l_{(a)} = l_{(b)} + k \). In the first case it follows that

\[
k(i - j) + l_{(b)} - l_{(a)} = k(i - j) \equiv 0 \mod 4k
\]

and thus \( i - j \geq 4 \) or \( a - b \geq 8 \). However, \( a - b \geq 8 \) implies that \( l_{(b)} \neq l_{(a)} \), a contradiction.

If \( l_{(a)} = l_{(b)} + k \), it follows that

\[
k(i - j) + l_{(b)} - l_{(a)} = k(i - j - 1) \equiv 0 \mod 4k,
\]

and thus \( 4 \) divides \( i - j - 1 \). We conclude that \( i - j - 1 \) is even and thus \( i - j \) is odd. It follows that \( i \) and \( j \) have different parities. But in this case \( p_3(\alpha_a) \) and \( p_3(\alpha_b) \) have different first coordinates, so \( p_3(\alpha_a) \neq p_3(\alpha_b) \). The argument when \( a \) and \( b \) are odd is similar.

Suppose then that \( a \) is even and \( b \) is odd. Considering the second coordinate of \( p_3(\alpha_a) - p_3(\alpha_b) \mod k \) gives that \( l_{(b)} - l_{(a)} \equiv 0 \mod k \). As \( |l_{(a)} - l_{(b)}| \leq 2k - 1 \),
we again conclude that \( l(a) = l(b) \) or \( l(a) = l(b) + k \). In the first case, considering the second coordinate of \( p_3(\alpha_a) - p_3(\alpha_b) \mod 2k \), we conclude \( k(i - j) \equiv 0 \mod 2k \), so \( (i - j) \geq 2 \). This however implies that \( l(b) \neq l(a) \), a contradiction. If \( l(a) = l(b) + k \), then, should the second coordinate of \( p_3(\alpha_a) - p_3(\alpha_b) \) be congruent to 0 \mod 4k, we’d have \( 2k(i - j - 1) \equiv 0 \mod 4k \), so \( (i - j - 1) \) is even. Again this shows that \( p_3(\alpha_a) \) and \( p_3(\alpha_b) \) have different first coordinates, so can not be equal. \( \square \)

**Lemma 21** The function \( c \circ p_3^{-1} : V(Z_{n,s}) \to N \) defines a radio labeling on \( Z_{4k,2} \).

**Proof.** As before it suffices to check all pairs of the form \((\alpha_{2i-1}, \alpha_{2i-1+a})\) and all pairs of the form \((\alpha_{2i}, \alpha_{2i+a})\) for \( a \leq 3 \). For the convenience of the reader, the values of \( p_3 \) for the pairs of vertices we must check are provided below.

| \( \alpha_{2i-1} \) | vertex \((i, 1 + k(i - 1) - l_i)\) | label value \((1 + (i - 1)(k + 1))\) |
| \( \alpha_{2i} \) | vertex \((i, 1 + k(i + 1) - l_i)\) | label value \((2 + (i - 1)(k + 1))\) |
| \( \alpha_{2(i+1)-1} \) | vertex \((i + 1, 1 + k - l_{i+1})\) | label value \((1 + i(k + 1))\) |
| \( \alpha_{2(i+1)} \) | vertex \((i + 1, 1 + k(i + 2) - l_{i+1})\) | label value \((2 + i(k + 1))\) |
| \( \alpha_{2(i+2)-1} \) | vertex \((i + 2, 1 + k(i + 1) - l_{i+2})\) | label value \((1 + (i + 1)(k + 1))\) |

We will have to compute distances in \( Z_{n,2} \). It is easy to see that \( d((1, j), (2, j')) = 1 \) and \( d((i, j), (i, j')) = \min\{|j - j'|, 4k - |j - j'|\} \). The distance \( d((1, j)(2, j')) \), \( j \neq j' \), is somewhat harder to compute. For this purpose we can use the standard cycle in \( Z_{4k,2} \) after appropriate renaming of the vertices. In particular, \( d((1, j)(2, j')) = d((1, j - j + 1), (2, j' - j + 1)) = d((1, 1), (2, j' - j + 1)). \) Let \( r \equiv j' - j + 1 \mod 4k \) and \( r \in \{1, \ldots, n\} \). Then \( d((1, j), (2, j')) = d((1, 1), (2, r)) = d_{C_{n+1}}(X_r^1, X_r^2) = \min\{r - 1, n + 1 - (r - 1)\} \). Note that in \( Z_{n,2} \), \( d((1, j)(2, j')) \neq d((1, j')(2, j)) \) thus \( d(\alpha_s, \alpha_t) \) depends on the parities of \( \left\lceil \frac{s}{2} \right\rceil \) and \( \left\lfloor \frac{s}{2} \right\rfloor \).
The following table shows the distances and label differences of the relevant pairs computed using the methods described above.

| vertex pair        | \(d(u, v), i \text{ even}\) | \(d(u, v), i \text{ odd}\) | \(|c(u) - c(v)|\) |
|--------------------|-------------------------------|-----------------------------|-----------------|
| \((\alpha_{2i-1}, \alpha_{2i})\) | 2\(k\)                       | 2\(k\)                      | 1               |
| \((\alpha_{2i-1}, \alpha_{2(i+1)-1})\) | \(d((1, 1), (2, 3k + 2))\) | \(d((1, 1)(2, k + 1))\) | \(k + 1\)      |
| \((\alpha_{2i-1}, \alpha_{2(i+1)})\) | \(= k\)                       | \(= k\)                     |                 |
| \((\alpha_{2i}, \alpha_{2(i+1)-1})\) | \(d((1, 1), (2, 3k + 1))\) | \(d((1, 1)(2, k + 1))\) | \(k\)          |
| \((\alpha_{2i}, \alpha_{2(i+1)})\) | \(= k + 1\)                   | \(= k + 1\)                 |                 |
| \((\alpha_{2i}, \alpha_{2(i+2)-1})\) | \(= k\)                       | \(= k\)                     | \(k + 1\)      |

\(\)(\n
Algorithm 4: \(n = 4k + 2\) when \(k\) is even and \(s = 3\)

The position function is

\[
p_4(\alpha_{2i-1}) = (l_i, 1 + (i - 1)k), \quad \text{and}\]
\[
p_4(\alpha_{2i}) = (l_i, 2 + (i + 1)k),\]

where

\[
l_i = \begin{cases} 
0, & i \leq 2k + 1, \\
1, & 2k + 1 < i \leq 4k + 2.
\end{cases}
\]

Lemma 22 The function \(p_4 : \{\alpha_j \mid j = 1, \ldots, 2n\} \to V(Z_{n,s})\) is a bijection.

PROOF.

Suppose \(p_4(\alpha_a) = p_4(\alpha_b)\). Let \(i = \left\lfloor \frac{a}{2} \right\rfloor\) and \(j = \left\lfloor \frac{b}{2} \right\rfloor\). If \(a\) and \(b\) have the same parity, it follows that \(ki \equiv kj \mod (4k + 2)\), i.e., \((i - j)k = (4k + 2)m\) for some integer \(m\). As \(k\) is even, \(k = 2q\) for some integer \(q\). Substituting and simplifying, we obtain the equation \(q(i - j) = m(4q + 1)\). As \(\gcd(q, 4q + 1) = 1\),
it follows that $q|m$ and thus $m \geq q$, so $(i - j) \geq 4q + 1 = 2k + 1$. But in this case $l_j \neq l_i$, so the first coordinates of $p_4(\alpha_a)$ and $p_4(\alpha_b)$ are different.

If $a$ is odd and $b$ is even, it follows that $1 + (j + 1)k - (i - 1)k \equiv 0 \mod 4k + 2$. So $1 + k(j - i + 2) \equiv 0 \mod 4k + 2$. As $k$ is even by hypothesis, $1 + k(j - i + 2)$ is odd, but $4k + 2$ is even, a contradiction. □

**Lemma 23** The function $c \circ p_4^{-1} : V(Z_{n,s}) \to \mathbb{N}$ defines a valid radio labeling on $Z_{4k+2,3}$ when $k$ is even.

**PROOF.** Since $\text{diam}(Z_{4k+2,3}) = 2k+1$, we need to show that $d(u, v) + |c(u) - c(v)| \geq 2k + 2$ for all pairs $u, v \in Z_{4k+2,3}$. Again it suffices to check only the pairs $(\alpha_{2i-1}, \alpha_{2i-1+a})$ and the pairs of the form $(\alpha_{2i}, \alpha_{2i+a})$ for $a \leq 3$. Below are given the positions and the labels of these vertices.

<table>
<thead>
<tr>
<th>vertex pair</th>
<th>label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2i})$</td>
<td>$1 + (i - 1)(k + 2)$</td>
</tr>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2(i+1)-1})$</td>
<td>$2 + (i - 1)(k + 2)$</td>
</tr>
<tr>
<td>$(\alpha_{2i}, \alpha_{2(i+1)})$</td>
<td>$1 + i(k + 2)$</td>
</tr>
<tr>
<td>$(\alpha_{2i-1}, \alpha_{2(i+1)-1})$</td>
<td>$2 + i(k + 2)$</td>
</tr>
<tr>
<td>$(\alpha_{2i}, \alpha_{2(i+2)-1})$</td>
<td>$1 + (i + 1)(k + 2)$</td>
</tr>
</tbody>
</table>

Note that in $Z_{n,3}$, $d((x_1, y_1), (x_2, y_2)) = \min\{|y_2 - y_1|, n - |y_1 - y_2|\}$ so the first coordinates of the vertices are irrelevant when computing distances. As $l_i$ only appears in the first coordinates, we do not have to consider the cases of $l_i = l_{i+1}$ and $l_i \neq l_{i+1}$ separately. Below are given all the relevant distances and label differences. It is easy to verify that the condition $d(u, v) + |c(u) - c(v)| \geq 2k + 2$ is satisfied for all pairs.

| vertex pair       | $d(u, v)$ | $|c(u) - c(v)|$ |
|-------------------|-----------|----------------|
| $(\alpha_{2i-1}, \alpha_{2i})$ | $2k+1$    | $1$           |
| $(\alpha_{2i-1}, \alpha_{2(i+1)-1})$ | $k$       | $k + 2$       |
| $(\alpha_{2i}, \alpha_{2(i+1)})$ | $1 + k$   | $k + 3$       |
| $(\alpha_{2i}, \alpha_{2(i+1)-1})$ | $k + 1$   | $k + 1$       |
| $(\alpha_{2i}, \alpha_{2(i+1)})$ | $k$       | $k + 2$       |
| $(\alpha_{2i}, \alpha_{2(i+2)-1})$ | $1$       | $2k + 3$      |

□
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Remark 16 By comparing values for each case we establish that in all cases considered by Algorithm 1

- \( \phi(n, s) + \omega \geq diam(Z_{n,s}) + 1 \) and that
- \( \phi(n, s) - \omega \geq \begin{cases} 
1, & \text{if } n - s \text{ is even} \\
2, & \text{if } n - s \text{ is odd.}
\end{cases} \)

First, we give the values of \( diam(Z_{n,s}) + 1 \):

\[
\begin{array}{c|c|c|c}
& s = 1 & s = 2 & s = 3 \\
\hline
r = 1 & 2k + 2 & 2k + 2 & 2k + 1 \\
\hline
r = 2 & 2k + 3 & 2k + 2 & 2k + 2 \\
\hline
r = 3 & 2k + 3 & 2k + 3 & 2k + 2 \\
\end{array}
\]

In each case, \( diam(Z_{n,s}) + 1 \leq \phi(n, s) + \omega \):

\[
\begin{array}{c|c|c|c}
\phi(n, s) + \omega & s = 1 & s = 2 & s = 3 \\
\hline
r = 1 & 2k + 2 & 2k + 2 & 2k + 1 \\
\hline
r = 2, k \text{ odd} & 2k + 3 & 2k + 2 & 2k + 2 \\
\hline
r = 2, k \text{ even} & 2k + 4 & 2k + 3 & \\
\hline
r = 3 & 2k + 3 & 2k + 4 & 2k + 3 \\
\end{array}
\]

The last table shows the values of \( \phi(n, s) - \omega \) with the entries corresponding to \( n + s \equiv 0 \mod 2 \) in bold.
\[
\phi(n, s) - \omega \\
\begin{array}{ccc}
  r = 1 & 2 & 2 \\
r = 2, k \text{ odd} & 3 & 2 \\
r = 2, k \text{ even} & 2 & 1 \\
r = 3 & 1 & 2 \\
\end{array}
\]

References


