A construction of complete-simple distributive lattices

George A. Menuhin*
Computer Science Department
University of Winnebago
Winnebago, Minnesota 53714

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Abstract

In this note we prove that there exist *complete-simple distributive lattices,* that is, complete distributive lattices in which there are only two complete congruences.

1 Introduction

In this note we prove the following result:

Theorem 1 *There exists an infinite complete distributive lattice K with only the two trivial complete congruence relations.*

2 The $D^{\langle 2 \rangle}$ construction

For the basic notation in lattice theory and universal algebra, see Ferenc R. Richardson [5] and George A. Menuhin [2]. We start with some definitions:

Definition 1 *Let V be a complete lattice, and let $\mathfrak{p} = [u, v]$ be* complete-prime *if the following three conditions are satisfied:*

1. *u is meet-irreducible but u is* not *completely meet-irreducible;*
2. *v is join-irreducible but v is* not *completely join-irreducible;*
3. *$[u, v]$ is a complete-simple lattice.*

Now we prove the following result:

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Lemma 1 *Let D be a complete distributive lattice satisfying conditions 1 and 2. Then $D^{\langle 2 \rangle}$ is a sublattice of D^2; hence $D^{\langle 2 \rangle}$ is a lattice, and $D^{\langle 2 \rangle}$ is a complete distributive lattice satisfying conditions 1 and 2.*

Proof. By conditions 1 and 2, $D^{\langle 2 \rangle}$ is a sublattice of D^2. Hence, $D^{\langle 2 \rangle}$ is a lattice.

Since $D^{\langle 2 \rangle}$ is a sublattice of a distributive lattice, $D^{\langle 2 \rangle}$ is a distributive lattice. Using the characterization of standard ideals in Ernest T. Moynahan [3], $D^{\langle 2 \rangle}$ has a zero and a unit element, namely, $\langle 0, 0 \rangle$ and $\langle 1, 1 \rangle$. To show that $D^{\langle 2 \rangle}$ is complete, let $\varnothing \neq A \subseteq D^{\langle 2 \rangle}$, and let $a = \bigvee A$ in D^2. If $a \in D^{\langle 2 \rangle}$, then $a = \bigvee A$ in $D^{\langle 2 \rangle}$; otherwise, a is of the form $\langle b, 1 \rangle$ for some $b \in D$ with $b < 1$. Now $\bigvee A = \langle 1, 1 \rangle$ in D^2 and the dual argument shows that $\bigwedge A$ also exists in D^2. Hence D is complete. Conditions 1 and 2 are obvious for $D^{\langle 2 \rangle}$. $\square$

Corollary 1 *If D is complete-prime, then so is $D^{\langle 2 \rangle}$.*

The motivation for the following result comes from Soo-Key Foo [1].

Lemma 2 *Let Θ be a complete congruence relation of $D^{\langle 2 \rangle}$ such that*

$$\langle 1, d \rangle \equiv \langle 1, 1 \rangle \pmod{\Theta}, \tag{1}$$

for some $d \in D$ with $d < 1$. Then $\Theta = \iota$.

Proof. Let Θ be a complete congruence relation of $D^{\langle 2 \rangle}$ satisfying (1). Then $\Theta = \iota$. $\square$

3 The Π^* construction

The following construction is crucial to our proof of Theorem 1:

Definition 2 *Let D_i, for $i \in I$, be complete distributive lattices satisfying condition 2. Their Π^* product is defined as follows:*

$$\Pi^*(D_i \mid i \in I) = \Pi(D_i^- \mid i \in I) + 1;$$

that is, $\Pi^*(D_i \mid i \in I)$ is $\Pi(D_i^- \mid i \in I)$ with a new unit element.

Figure 1 illustrates this construction.

Notation 1 *If $i \in I$ and $d \in D_i^-$, then*

$$\langle \ldots, 0, \ldots, \overset{i}{d}, \ldots, 0, \ldots \rangle$$

is the element $\langle f(j) \rangle_{j \in I}$ of $\Pi^*(D_i \mid i \in I)$ defined by

$$f(j) = \begin{cases} d, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

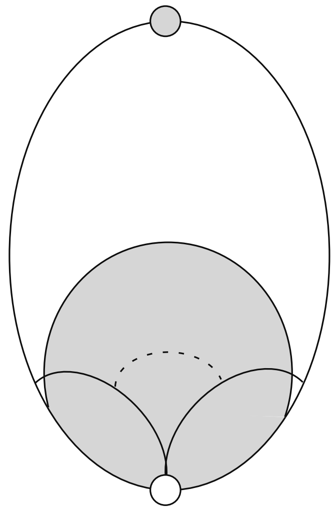

Figure 1: Illustrating $\Pi^*(D_i \mid i \in I)$ in $\Pi(D_i \mid i \in I)$.

See also Ernest T. Moynahan [4]. Next we verify:

Theorem 2 *Let D_i, for $i \in I$, be complete distributive lattices satisfying condition 2. Let Θ be a complete congruence relation on $\Pi^*(D_i \mid i \in I)$. If there exist $i \in I$ and $d \in D_i$ with $d < 1_i$ such that for all $d \leq c < 1_i$,*

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \equiv \langle \dots, 0, \dots, \overset{i}{c}, \dots, 0, \dots \rangle \pmod{\Theta}, \tag{2}$$

then $\Theta = \iota$.

Proof. Since

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \equiv \langle \dots, 0, \dots, \overset{i}{c}, \dots, 0, \dots \rangle \pmod{\Theta}, \tag{3}$$

and Θ is a complete congruence relation, meeting both sides of the congruence (3) with $\langle \dots, 0, \dots, \overset{j}{a}, \dots, 0, \dots \rangle$, we obtain

$$\begin{aligned} 0 &= \langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \wedge \langle \dots, 0, \dots, \overset{j}{a}, \dots, 0, \dots \rangle \\ &\equiv \langle \dots, 0, \dots, \overset{j}{a}, \dots, 0, \dots \rangle \pmod{\Theta}. \end{aligned} \tag{4}$$

Using the completeness of Θ and (4), we get:

$$0 \equiv \bigvee (\langle \dots, 0, \dots, \overset{j}{a}, \dots, 0, \dots \rangle \mid a \in D_j^-) = 1 \pmod{\Theta},$$

hence $\Theta = \iota$. $\square$

Theorem 3 *Let D_i for $i \in I$ be complete distributive lattices satisfying conditions 2 and 3. Then $\Pi^*(D_i \mid i \in I)$ also satisfies conditions 2 and 3.*

Proof. Let Θ be a complete congruence on $\Pi^*(D_i \mid i \in I)$. Let $i \in I$. Define

$$\widehat{D}_i = \{\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \mid d \in D_i^- \} \cup \{1\}.$$

Then $\widehat{D}_i$ is a complete sublattice of $\Pi^*(D_i \mid i \in I)$, and $\widehat{D}_i$ is isomorphic to D_i. Let Θ_i be the restriction of Θ to $\widehat{D}_i$.

Since D_i is complete-simple, so is $\widehat{D}_i$, and hence Θ_i is ω or ι. If $\Theta_i = \rho$ for all $i \in I$, then $\Theta = \omega$. If there is an $i \in I$, such that $\Theta_i = \iota$, then $0 \equiv 1 \pmod{\Theta}$, hence $\Theta = \iota$. $\square$

Theorem 1 follows easily from Theorems 2 and 3.

References

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