A construction of complete-simple distributive lattices

George A. Menuhin*
Computer Science Department
University of Winnebago
Winnebago, Minnesota 53714

September 15, 2000

Abstract
In this note we prove that there exist complete-simple distributive lattices, that is, complete distributive lattices in which there are only two complete congruences.

1 Introduction
In this note we prove the following result:

Theorem 1 There exists an infinite complete distributive lattice $K$ with only the two trivial complete congruence relations.

2 The $D^{(2)}$ construction
For the basic notation in lattice theory and universal algebra, see Ferenc R. Richardson [5] and George A. Menuhin [2]. We start with some definitions:

Definition 1 Let $V$ be a complete lattice, and let $p = [u, v]$ be complete-prime if the following three conditions are satisfied:

1. $u$ is meet-irreducible but $u$ is not completely meet-irreducible;
2. $v$ is join-irreducible but $v$ is not completely join-irreducible;
3. $[u, v]$ is a complete-simple lattice.

Now we prove the following result:

*Research supported by the NSF under grant number 23466.
Lemma 1 Let $D$ be a complete distributive lattice satisfying conditions 1 and 2. Then $D^{(2)}$ is a sublattice of $D^2$; hence $D^{(2)}$ is a lattice, and $D^{(2)}$ is a complete distributive lattice satisfying conditions 1 and 2.

Proof. By conditions 1 and 2, $D^{(2)}$ is a sublattice of $D^2$. Hence, $D^{(2)}$ is a lattice.

Since $D^{(2)}$ is a sublattice of a distributive lattice, $D^{(2)}$ is a distributive lattice. Using the characterization of standard ideals in Ernest T. Moynahan [3], $D^{(2)}$ has a zero and a unit element, namely, $\langle 0,0 \rangle$ and $\langle 1,1 \rangle$. To show that $D^{(2)}$ is complete, let $\emptyset \neq A \subseteq D^{(2)}$, and let $a = \bigvee A$ in $D^2$. If $a \in D^{(2)}$, then $a = \bigvee A$ in $D^{(2)}$; otherwise, $a$ is of the form $\langle b,1 \rangle$ for some $b \in D$ with $b < 1$. Now $\bigvee A = \langle 1,1 \rangle$ in $D^2$ and the dual argument shows that $\bigwedge A$ also exists in $D^2$. Hence $D$ is complete. Conditions 1 and 2 are obvious for $D^{(2)}$. □

Corollary 1 If $D$ is complete-prime, then so is $D^{(2)}$.

The motivation for the following result comes from Soo-Key Foo [1].

Lemma 2 Let $\Theta$ be a complete congruence relation of $D^{(2)}$ such that

$$\langle 1,d \rangle \equiv \langle 1,1 \rangle \pmod{\Theta},$$

for some $d \in D$ with $d < 1$. Then $\Theta = \iota$.

Proof. Let $\Theta$ be a complete congruence relation of $D^{(2)}$ satisfying (1). Then $\Theta = \iota$. □

3 The $\Pi^*$ construction

The following construction is crucial to our proof of Theorem 1:

Definition 2 Let $D_i$, for $i \in I$, be complete distributive lattices satisfying condition 2. Their $\Pi^*$ product is defined as follows:

$$\prod^*(D_i \mid i \in I) = \prod(D_i^\ast \mid i \in I) + 1;$$

that is, $\prod^*(D_i \mid i \in I)$ is $\prod(D_i^\ast \mid i \in I)$ with a new unit element.

Figure 1 illustrates this construction.

Notation 1 If $i \in I$ and $d \in D_i^\ast$, then

$$\langle \ldots, 0, \ldots, i_d, \ldots, 0, \ldots \rangle$$

is the element $\langle f(j) \rangle_{j \in I}$ of $\prod^*(D_i \mid i \in I)$ defined by

$$f(j) = \begin{cases} d, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$
See also Ernest T. Moynahan [4]. Next we verify:

**Theorem 2** Let $D_i$, for $i \in I$, be complete distributive lattices satisfying condition 2. Let $\Theta$ be a complete congruence relation on $\Pi^*(D_i \mid i \in I)$. If there exist $i \in I$ and $d \in D_i$ with $d < 1_i$ such that for all $d \leq c < 1_i$,

$$
\langle \ldots, 0, \ldots, d, \ldots, 0, \ldots \rangle \equiv \langle \ldots, 0, \ldots, c, \ldots, 0, \ldots \rangle \quad \text{(mod } \Theta) \quad \text{ (2)}
$$

then $\Theta = \iota$.

**Proof.** Since

$$
\langle \ldots, 0, \ldots, \hat{i}, \ldots, 0, \ldots \rangle \equiv \langle \ldots, 0, \ldots, \hat{i}, \ldots, 0, \ldots \rangle \quad \text{(mod } \Theta) \quad \text{ (3)}
$$

and $\Theta$ is a complete congruence relation, meeting both sides of the congruence (3) with $\langle \ldots, 0, \ldots, \hat{j}, \ldots, 0, \ldots \rangle$, we obtain

$$
0 = \langle \ldots, 0, \ldots, \hat{i}, \ldots, 0, \ldots \rangle \land \langle \ldots, 0, \ldots, \hat{j}, \ldots, 0, \ldots \rangle \quad \text{ (4)}
$$

$$
\equiv \langle \ldots, 0, \ldots, \hat{j}, \ldots, 0, \ldots \rangle \quad \text{(mod } \Theta).
$$

Using the completeness of $\Theta$ and (4), we get:

$$
0 \equiv \bigvee \{\langle \ldots, 0, \ldots, \hat{j}, \ldots, 0, \ldots \rangle \mid a \in D_\emptyset \} = 1 \quad \text{(mod } \Theta),
$$

hence $\Theta = \iota$. $\square$
Theorem 3 Let $D_i$ for $i \in I$ be complete distributive lattices satisfying conditions 2 and 3. Then $\Pi^*(D_i \mid i \in I)$ also satisfies conditions 2 and 3.

Proof. Let $\Theta$ be a complete congruence on $\Pi^*(D_i \mid i \in I)$. Let $i \in I$. Define

$$\hat{D}_i = \{\langle \ldots, 0, \ldots, d, \ldots, 0, \ldots \rangle \mid d \in D_i^- \} \cup \{1\}.$$ 

Then $\hat{D}_i$ is a complete sublattice of $\Pi^*(D_i \mid i \in I)$, and $\hat{D}_i$ is isomorphic to $D_i$. Let $\Theta_i$ be the restriction of $\Theta$ to $\hat{D}_i$.

Since $D_i$ is complete-simple, so is $\hat{D}_i$, and hence $\Theta_i$ is $\omega$ or $\iota$. If $\Theta_i = \rho$ for all $i \in I$, then $\Theta = \omega$. If there is an $i \in I$, such that $\Theta_i = \iota$, then $0 \equiv 1 \,(\text{mod } \Theta)$, hence $\Theta = \iota$. $\square$

Theorem 1 follows easily from Theorems 2 and 3.

References


