YOUNG TABLEAUX, WEBS, AND THE SYMMETRIC GROUP
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Given a natural number $n \in \mathbb{N}$, a partition $\lambda \vdash n$ is a decomposition of $n$ into an increasing sum of natural numbers. For example, associated to the sum $3+2+2+1 = 8$ we have the partition $\lambda = (3, 2, 2, 1) \vdash 8$. One way mathematicians study partitions is via Young diagrams which are collections of $n$ top and left justified boxes with rows corresponding to the elements of $\lambda$. Figure 1 shows an example of a Young diagram.

Young tableaux are fillings of Young diagrams with numbers. In our project, we will be interested in standard and semi-standard Young tableaux. Standard tableaux are fillings with the numbers 1 through $n$ such that rows increase from left to right and columns increase from top to bottom. Semi-standard tableaux are fillings with the same increasing row and column property, however they are allowed to contain repeated numbers. Examples of standard and semi-standard fillings are also given in Figure 1.

![Young diagram](image1)

Figure 1. The Young diagram for $\lambda = (3, 3, 1) \vdash 7$ together with standard and semi-standard Young tableaux of shape $\lambda$

Young tableaux are basic combinatorial objects that appear throughout mathematics. We will be interested in their connection to $sl_2$ and $sl_3$ webs, which are certain types of planar graphs that arise in knot theory and representation theory [3]. Given some $n, k \in \mathbb{N}$ such that $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, an $sl_2$ web (or crossingless matching) of type $(n - k, k)$ is a nonintersecting arrangement of $k$ semi-circular arcs and $n - 2k$ rays incident on $n$ horizontally aligned vertices. An $sl_3$ web is a directed planar graph with internal and external vertices. All internal vertices must be trivalent, all external vertices must be univalent, and every vertex must be either a source or a sink. An $sl_3$ web is said to be irreducible if it has no bigon or square faces. Examples of webs are shown in Figure 2.

![Web](image2)

Figure 2. An $sl_2$ web of type $(3, 1)$ and an irreducible $sl_3$ web with 9 external source vertices with their associated standard tableaux
Project 1: Semi-standard tableaux and $sl_3$ webs

In [6] we give an explicit bijection between $sl_2$ webs of type $(n-k, k)$ and standard Young tableaux of shape $(n-k, k)$. There is also an explicit bijection between $sl_3$ webs with $3n$ external source vertices and standard Young tableaux of shape $(n, n, n)$ [7]. Figure 2 shows examples of these bijections. For $n, k \in \mathbb{N}$, it has been shown that the number of $sl_3$ webs with $k$ source and $3n-2k$ sink external vertices is the same as the number of semi-standard Young tableaux filled with the elements of the set $\{1, 1, 2, 2, \ldots, k, k, k+1, \ldots, 3n-k\}$ [5]. In this project, students will work on finding an explicit bijection between these two sets that generalizes the one given in [7]. In other words, they will come up with an algorithm for constructing a unique web corresponding to each tableau and vice versa.

Note: For this project, some experience with proof-writing would be helpful.

Project 2: The permutation action on $sl_2$ webs

A permutation on an ordered set of objects is a rearrangement of those objects. The collection of permutations on an ordered set of size $n$ forms a group known as the symmetric group on $n$ letters, usually denoted $S_n$. The symmetric group $S_{2n}$ acts on $sl_2$ webs of type $(n, n)$ [5]. In other words, permutations induce maps on the set of these webs. The symmetric group also acts on a certain collection of polynomials in $2n$ variables which we will call the Garsia-Procesi polynomials [2]. The Garsia-Procesi polynomials are in bijection with standard Young tableaux of shape $(n, n)$ and are thus also in bijection with $sl_2$ webs of type $(n, n)$ [4]. In fact the symmetric group actions on Garsia-Procesi polynomials and webs are actually the same (in the sense of representation theory). The goal for this project is to find a map between webs and Garsia-Procesi polynomials that commutes with the symmetric group action. Such a map is called an equivariant map.

Note: For the second project, a familiarity with linear algebra and basic group theory would be immensely helpful.

References