

# THE RADIO NUMBER OF GEAR GRAPHS

CHRISTINA FERNANDEZ, AMÉRICA FLORES, MAGGY TOMOVA,  
AND CINDY WYELS

ABSTRACT. Let  $d(u, v)$  denote the distance between two distinct vertices of a connected graph  $G$ , and  $\text{diam}(G)$  be the diameter of  $G$ . A *radio labeling*  $c$  of  $G$  is an assignment of positive integers to the vertices of  $G$  satisfying  $d(u, v) + |c(u) - c(v)| \geq \text{diam}(G) + 1$ . The maximum integer in the range of the labeling is its span. The *radio number* of  $G$ ,  $rn(G)$ , is the minimum possible span. The family of gear graphs of order  $n$ ,  $G_n$ , consists of planar graphs with  $2n + 1$  vertices and  $3n$  edges. We prove that the radio number of the  $n$ -gear is  $4n + 2$ .

**2000 AMS Subject Classification:** 05C78 (05C15)

## 1. INTRODUCTION

Radio labeling of graphs is motivated by restrictions inherent in assigning channel frequencies for radio transmitters [1]. To avoid interference, transmitters that are geographically close must be assigned channels with large frequency differences; transmitters that are further apart may receive channels with relatively close frequencies. The general situation is modeled by identifying transmitters with the vertices of a graph. We assign positive integers to the vertices of the graph subject to a restriction concerning the distances between vertices; the goal is to minimize the largest integer used.

We will consider simple connected graphs  $G = (V(G), E(G))$ . We write  $d(u, v)$  for the distance between vertices  $u$  and  $v$ , and use  $\text{diam}(G)$  to indicate the diameter of  $G$ . A *radio labeling* is a one-to-one mapping  $c : V(G) \rightarrow \mathbf{Z}_+$  satisfying the condition

$$(1) \quad d(u, v) + |c(u) - c(v)| \geq \text{diam}(G) + 1$$

for every  $u, v \in V(G)$ . The *span* of a labeling  $c$  is the maximum integer that  $c$  maps to a vertex of graph  $G$ . The *radio number* of  $G$ ,  $rn(G)$ , is the lowest span taken over all radio labelings of the graph

---

*Date:* September 12, 2008.

*Key words and phrases.* radio number, radio labeling, gear graph.

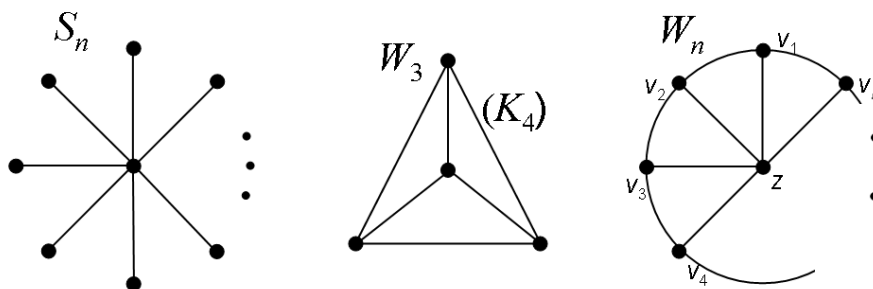


FIGURE 1

$G^1$ . We will refer to Inequality 1 as *the radio condition*. Note that this condition necessitates the use of distinct integers, thus  $rn(G) \geq |V(G)|$  for all graphs  $G$ . Radio labelings are sometimes referred to as multi-distance labelings (e.g. [2]), and they are equivalent to  $k$ -labelings for  $k = \text{diam}(G)$ .

In this introduction we will briefly note the radio numbers of some common families of graphs: complete graphs, stars, and wheels. The main section is devoted to establishing the radio number of gear graphs, which are extensions of wheel graphs.

**Theorem 1.1.** *The radio number of the complete graph on  $n$  vertices is  $n$ , i.e.  $rn(K_n) = n$ .*

*Proof.* Since  $|V(K_n)| = n$ , we have  $rn(K_n) \geq n$ . As  $\text{diam}(K_n) = 1$ , we may satisfy the radio condition while labeling the vertices with consecutive integers.  $\square$

The star graph  $S_n$  ( $n \geq 2$ ) is a tree on  $n+1$  vertices. One vertex (the “center”) is adjacent to every vertex; all other vertices have degree 1. That is,  $S_n = K_{1,n}$ . (See Figure 1.)

**Theorem 1.2.**  $rn(S_n) = n + 2$ .

*Proof.* Note that  $\text{diam}(S_n) = 2$ . This together with the fact that the center vertex is adjacent to every other vertex implies we may not use consecutive integers to label the center and another vertex. Since  $|V(S_n)| = n + 1$ , we see  $rn(S_n) \geq n + 2$ . Assigning 1 to the center and consecutive integers beginning with 3 to the other vertices produces a radio labeling with span  $n + 2$ , so  $rn(S_n) = n + 2$ .  $\square$

**Remark 1.3.**  $rn(K_{m,n}) = m + n + 1$ .

<sup>1</sup>Some authors allow 0 in the range of a radio labeling (e.g. [2]); their radio numbers will be one less than what we define.

*Proof.* Using similar logic as in the proof of Theorem 1.2, we see that  $rn(K_{m,n}) \geq m + n + 1$ . This lower bound may be realized by the span of a radio labeling: use the label set  $\{1, 2, \dots, m\}$  on the partition of size  $m$ , and the label set  $\{m + 2, m + 3, \dots, m + n + 1\}$  on the other partition.  $\square$

The *wheel graph*  $W_n$  ( $n \geq 3$ ) consists of an  $n$ -cycle together with a center vertex that is adjacent to all  $n$  vertices of the cycle. (See Figure 1.)  $W_3 = K_4$ , so  $rn(W_3) = 4$ . As  $\text{diam}(W_4) = 2$ , adjacent vertices must have non-consecutive labels. There are only two mutually-exclusive pairs of non-adjacent vertices, so at most two pairs of consecutive numbers may be used to label the five vertices. This gives a lower bound for  $rn(W_4)$  of 7; a labeling assigning 1 to the center vertex and 3, 6, 4, and 7 (sequentially) to the vertices on the cycle realizes this bound and is a radio labeling. Thus  $rn(W_4) = 7$ . The radio number of all larger wheels is given in Theorem 1.4.

**Theorem 1.4.**  $rn(W_n) = n + 2$  for  $n \geq 5$ .

*Proof.* For  $n \geq 4$ ,  $\text{diam}(W_n) = 2$ . As with the star, we may not use consecutive integers to label the center and another vertex, so  $rn(W_n) \geq n + 2$ . To define a radio labeling, we name the center vertex  $z$ , and identify the remaining vertices as  $\{v_1, v_2, \dots, v_n\}$ , labeled sequentially around the cycle (as shown in Figure 1). Our labeling  $c$  assigns  $z$  the label 1. Vertices  $v_1, v_2, \dots, v_{\lceil \frac{n}{2} \rceil}$  are labeled with consecutive odd numbers, beginning with 3. Vertices  $v_{\lceil \frac{n}{2} \rceil + 1}, \dots, v_n$  are assigned consecutive even integers beginning with 4.

We must show that the radio condition is satisfied for all pairs of vertices. Consider the pair  $(z, v_i)$ , noting that  $d(z, v_i) = 1$ . We have  $c(z) = 1$  and  $c(v_i) \geq 3$ . So

$$d(z, v_i) + |c(z) - c(v_i)| \geq 1 + |3 - 1| \geq 3 = \text{diam}(W_n) + 1.$$

If  $v_i$  and  $v_j$  ( $i \neq j$ ) are adjacent, their labels differ by at least 2. Otherwise,  $d(v_i, v_j) \geq 2$ , and  $|c(v_i) - c(v_j)| \geq 1$ . In either case, the radio condition is satisfied.

As the range of  $c$  is the label set  $\{1, 3, 4, \dots, n + 2\}$ , this establishes  $rn(W_n) = n + 2$ .  $\square$

## 2. THE RADIO NUMBER OF GEAR GRAPHS

The proof of Theorem 1.4 models the strategy we will use to establish the radio number of gear graphs. The lower bound is found by examining the minimum necessary differences between labels. To determine an upper bound, we provide a specific radio labeling. As the labeling

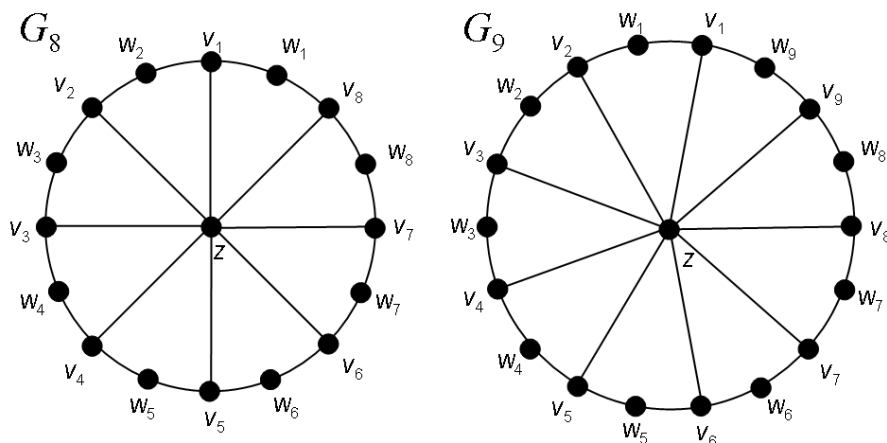


FIGURE 2

provided has span equal to the lower bound, their common value is the radio number.

Gear graphs are extensions of wheel graphs. We may describe the  $n$ -gear,  $G_n$ , as  $W_n$  with an additional vertex inserted between each pair of vertices on the cycle. Equivalently, the  $n$ -gear consists of a cycle on  $2n$  vertices, with every other vertex on the cycle adjacent to an additional  $2n + 1^{\text{st}}$  vertex (the center). Gear graphs have  $3n$  edges. For  $n \geq 4$ ,  $\text{diam}(G_n) = 4$ .

*The Standard Labeling of the  $n$ -Gear*

To establish the radio number of the  $n$ -gear we will refer to a labeling of the vertices of the  $n$ -gear that distinguishes the vertices by their characteristics. The center vertex is labeled  $z$ , and the vertices adjacent to the center are labeled sequentially  $\{v_1, v_2, \dots, v_n\}$ . The vertices not adjacent to the center are labeled sequentially  $\{w_1, w_2, \dots, w_n\}$ , using the same orientation chosen for the  $v$ 's. If  $n$  is odd then we specify that  $w_1$  is adjacent to  $v_1$  and  $v_2$ , otherwise  $w_1$  is adjacent to  $v_1$  and  $v_n$ . The standard labelings of the 8-gear and of the 9-gear are shown in Figure 2.

**Theorem 2.1.** *For  $n \geq 4$ ,  $rn(G_n) \geq 4n + 2$ .*

*Proof.* Assume  $n \geq 4$ . This gives  $\text{diam}(G_n) = 4$ , so any radio labeling  $c$  of  $G_n$  must satisfy the radio condition

$$d(u, v) + |c(u) - c(v)| \geq 5$$

for all distinct  $u, v \in V(G)$ . We count the number of values needed for labels and add the minimum number of *forbidden* values – those values

precluded by use of a particular value as a label. For instance, if we label the center  $a$  (i.e.  $c(z) = a$ ), then, as  $d(z, r) \leq 2$  for all vertices  $r \neq z$ , the values  $a - 2$ ,  $a - 1$ ,  $a + 1$ , and  $a + 2$  are forbidden. Similarly, as  $d(v_i, r) \leq 3$  for all  $v_i$  and for any  $r \neq v_i$ , one value is forbidden above and below any label  $c(v_i)$ . However, as  $d(w_i, r) = 4$  for some vertex  $r$ , it is possible to use consecutive labels on  $w_i$  and  $r$ . (I.e. there are no forbidden values associated with the vertices  $\{w_1, \dots, w_n\}$ .)

Note that the number of forbidden values is symmetric below and above any label used for a particular vertex. Thus we find the minimum number of forbidden values by using the lowest and highest-valued labels on the center vertex and on one of  $\{v_1, \dots, v_n\}$ . Assume without loss of generality that  $c(z) = 1$  and  $c(v_n)$  is the span of  $c$ . Associated with the center vertex are then two forbidden values (2 and 3), with  $v_n$  is one forbidden value (the span less one), and with the other  $v_i$  are two forbidden values each. This gives a total of  $2 + 1 + 2(n - 1) = 2n + 1$  forbidden values. Adding in the  $2n + 1$  values needed to label the  $2n + 1$  vertices provides a total of  $4n + 2$ , hence  $rn(G_n) \geq 4n + 2$ .  $\square$

**Theorem 2.2.** *For  $n \geq 7$ ,  $rn(G_n) \leq 4n + 2$ .*

*Proof.* We provide a radio labeling  $c$  of  $G_n$  in two steps: first we define a position function that renames the vertices of  $G_n$  using the set  $\{x_0, x_1, \dots, x_{2n}\}$ , then we specify the labels  $c(x_i)$  so that  $i < j$  if and only if  $c(x_i) < c(x_j)$ . (This allows us to more easily show that  $c$  is indeed a radio labeling.) Once it is established that  $c$  is a radio labeling, the span of  $c$  provides an upper bound for  $rn(G_n)$ . Throughout this proof,  $n \geq 7$ .

The position function  $p : V(G_n) \rightarrow \{x_0, x_1, \dots, x_{2n}\}$  is defined as follows. For  $n = 2k + 1$  we define

$$\begin{aligned} p(z) &= x_0, \\ p(w_{2i-1}) &= x_i && \text{for } i = 1, \dots, k + 1, \\ p(w_{2i}) &= x_{k+1+i} && \text{for } i = 1, \dots, k, \\ p(v_i) &= x_{n+i} && \text{for } i = 1, \dots, n. \end{aligned}$$

When  $n = 2k$  the position function changes slightly in renaming the vertices  $w_i$ :

$$\begin{aligned} p(z) &= x_0, \\ p(w_{2i-1}) &= x_i && \text{for } i = 1, \dots, k, \\ p(w_{2i}) &= x_{k+i} && \text{for } i = 1, \dots, k, \\ p(v_i) &= x_{n+i} && \text{for } i = 1, \dots, n. \end{aligned}$$

In essence, the position function orders the vertices so that  $\{x_0, x_1, \dots, x_{2n}\}$  corresponds to  $\{z, w_1, w_3, \dots, w_n, w_2, w_4, \dots, w_{n-1}, v_1, v_2, \dots, v_n\}$  when

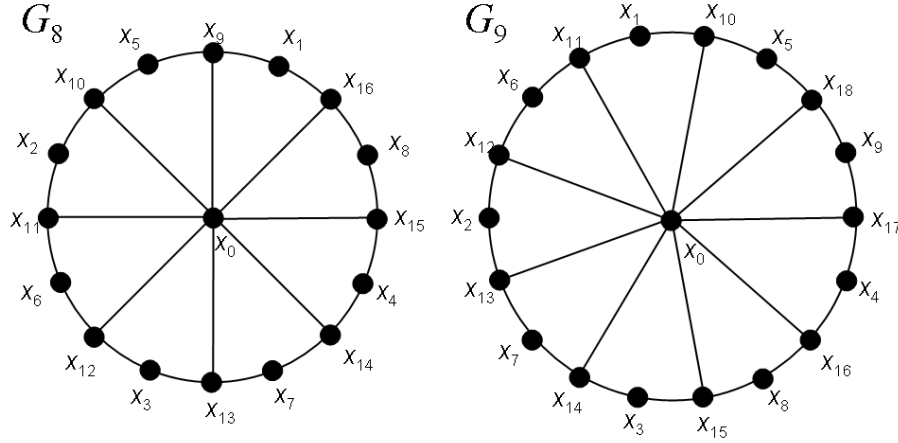


FIGURE 3

$n$  is odd and to  $\{z, w_1, w_3, \dots, w_{n-1}, w_2, w_4, \dots, w_n, v_1, v_2, \dots, v_n\}$  when  $n$  is even. Figure 3 shows the renamed versions of the 8-gear and the 9-gear.

We are now ready to define our radio labeling  $c : V(G) \rightarrow \mathbf{Z}_+$ .

$$c(x_i) = \begin{cases} 1, & i = 0, \\ 3 + i, & 1 \leq i \leq n, \\ n + 2 + 3(i - n), & n + 1 \leq i \leq 2n. \end{cases}$$

Figure 3 shows the labeling  $c$  applied to the 8-gear and to the 9-gear.

**Claim:** The labeling  $c$  is a valid radio labeling. We must show that the radio condition,

$$d(u, v) + |c(u) - c(v)| \geq \text{diam}(G_n) + 1 = 5,$$

holds for all pairs of vertices  $(u, v)$  (where  $u \neq v$ ).

**Case 1:** Consider the pair  $(z, r)$  (for any vertex  $r \neq z$ ).

Recall  $p(z) = x_0$ . As  $c(x_i) \geq 5$  for any  $i \geq 2$ , we have  $d(z, x_i) + |c(z) - c(x_i)| \geq 1 + |1 - 5| \geq 5$  for all  $i \geq 2$ . This leaves the pair  $(z, x_1)$ . But  $p^{-1}(x_1) = w_1$ , so we calculate  $d(z, w_1) + |c(z) - c(w_1)| = 2 + |1 - 4| = 5$ .

**Case 2:** Consider the pair  $(w_j, w_k)$  (with  $j \neq k$ ).

Recall  $p(w_{2i-1}) = x_i$  and note that  $p(w_{2i})$  may be written as  $x_{n-k+i}$  whether  $n$  is even or odd. We have  $d(w_j, w_k) = 2$  for the pairs  $(w_{2i-1}, w_{2i})$ ,  $(w_{2i}, w_{2i+1})$ , and  $(w_n, w_1)$ . These “translate”, respectively, to  $(x_i, x_{n-k+i})$ ,  $(x_{n-k+i}, x_{i+1})$ , and  $(x_s, x_1)$ , where  $s = k + 1$  when  $n$  is odd and  $s = 2k$  when  $n$  is even. Examine the label difference for each pair:  $|c(x_i) - c(x_{n-k+i})| = n - k$ ,  $|c(x_{n-k+i}) - c(x_{i+1})| = n - k - 1$ , and  $|c(x_s) - c(x_1)|$  is  $k$  when  $s = k + 1$  ( $n$  odd) and is  $2k - 1$  when  $s = 2k$  ( $n$  even). In all cases, using the fact that  $n \geq 7$ , we have

that  $|c(w_j) - c(w_k)| \geq 3$ , so the radio condition is satisfied whenever  $d(w_j, w_k) = 2$ . Meanwhile, should  $j$  and  $k$  not be consecutive (mod  $n$ ), we have  $d(w_j, w_k) \geq 4$ . As it is always the case that  $|c(w_j) - c(w_k)| \geq 1$ , the radio condition is again satisfied.

**Case 3:** Examine the pair  $(v_j, v_k)$  (with  $j \neq k$ ).

Note that  $d(v_j, v_k) = 2$ . Since  $|c(v_j) - c(v_k)| = |c(x_{n+j}) - c(x_{n+k})| \geq 3$  for all  $v_j, v_k$ , the radio condition is satisfied.

**Case 4:** Finally, consider the pair  $(v, w)$ , where  $v \in \{v_1, \dots, v_n\}$  and  $w \in \{w_1, \dots, w_n\}$ .

We have  $c(v) \in \{n + 5, n + 8, \dots, 4n + 2\}$  and  $c(w) \in \{4, 5, 6, \dots, n + 3\}$ . For all  $v \neq v_1$ ,  $|c(v) - c(w)| \geq (n + 8) - (n + 3) = 5$ . Therefore the radio condition is satisfied when  $v \neq v_1$ . Meanwhile,  $|c(v_1) - c(w)| \geq (n + 5) - (n + 3) = 2$ . If  $w$  is distance three or greater from  $v_1$ , the radio condition holds. If  $d(v_1, w) < 3$ , then  $w = x_1$  or  $w = x_{\lfloor \frac{n}{2} \rfloor + 1}$ . Checking the radio condition for each we obtain  $d(v_1, x_1) + |c(v_1) - c(x_1)| = 1 + |(n + 5) - 4| = n + 2 \geq 5$ , and  $d(v_1, x_{\lfloor \frac{n}{2} \rfloor + 1}) + |c(v_1) - c(x_{\lfloor \frac{n}{2} \rfloor + 1})| = 1 + |(n + 5) - (3 + \lfloor \frac{n}{2} \rfloor + 1)| \geq \frac{n}{2} + 3 \geq 5$ .

These four cases establish the claim that  $c$  is a radio labeling of  $G_n$ . Thus  $rn(G_n) \leq \text{span}(c) = c(2n) = n + 2 + 3(2n - n) = 4n + 2$ .  $\square$

Taken together, Theorem 2.1 and Theorem 2.2 establish the main result of this paper:

**Theorem 2.3.** *The radio number of the  $n$ -gear is  $4n + 2$  when  $n \geq 4$ .*

*Proof.* Theorem 2.1 shows  $rn(G_n) \geq 4n + 2$  for  $n \geq 4$ , and Theorem 2.2 shows  $rn(G_n) \leq 4n + 2$  for  $n \geq 7$ . It remains only to show that  $rn(G_n) \leq 4n + 2$  for  $n = 4, 5, 6$ . This is demonstrated by the radio labelings provided in Figure 4. Note that the labels with values  $a$  and  $a + 2$  are assigned to vertices at distance 3.  $\square$

For completeness' sake, we provide radio labelings of  $G_n$  for  $n = 2, \dots, 6$  in Figure 4. The reader may verify that each radio labeling provided uses the minimum possible span, and thus exhibits the radio number of each small gear.

### 3. ACKNOWLEDGEMENTS

This research was carried out under the auspices of an MAA (SUMMA) Research Experience for Undergraduates program funded by NSA, NSF, and Moody's, and hosted at CSU Channel Islands during Summer, 2006. We are grateful to all for the opportunities provided.

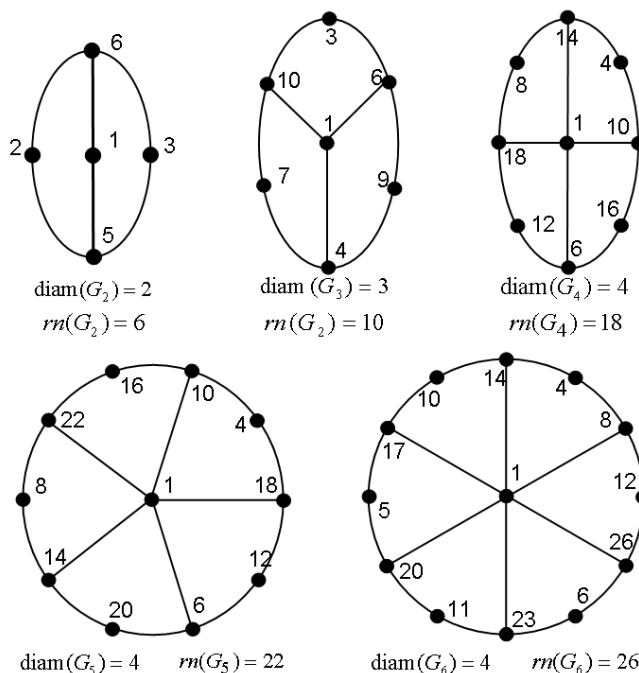


FIGURE 4

REFERENCES

[1] Chartrand, Gary and Erwin, David and Zhang, Ping and Harary, Frank, *Radio labelings of graphs*, Bull. Inst. Combin. Appl., **33** (2001), 77–85.  
 [2] Liu, Daphne Der-Fen and Zhu, Xuding, *Multilevel distance labelings for paths and cycles*, SIAM J. Discrete Math. **19** (2005), No. 3, 610–621.

DEPARTMENT OF MATHEMATICS, CSU CHANNEL ISLANDS  
*E-mail address:* ma.christy@teacher.com

DEPARTMENT OF MATHEMATICS, CSU CHANNEL ISLANDS  
*E-mail address:* america.flores600@dolphin.csuci.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA  
*E-mail address:* mtomova@math.uiowa.edu

DEPARTMENT OF MATHEMATICS, CSU CHANNEL ISLANDS  
*E-mail address:* cindy.wyels@csuci.edu