THE RADIO NUMBER OF $C_n \square C_n$

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Abstract. Radio labeling is a variation of Hale's channel assignment problem, in which one seeks to assign positive integers to the vertices of a graph G subject to certain constraints involving the distances between the vertices. Specifically, a radio labeling of a connected graph G is a function $c : V(G) \to \mathbf{N}$ such that

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(G)$$

for every two distinct vertices u and v of G. The span of a radio labeling is the maximum integer assigned to a vertex. The radio number of a graph G is the minimum span, taken over all radio labelings of G. This paper establishes the radio number of the Cartesian product of a cycle graph with itself (i.e. of $C_n \square C_n$.)

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1. Introduction

Radio labeling is derived from the assignment of radio frequencies (channels) to a set of transmitters. The frequencies assigned depend on the geographical distance between the transmitters: the closer two transmitters are, the greater the potential for interference between their signals. Thus when the distance between two transmitters is small, the difference in the frequencies assigned must be relatively large, whereas two transmitters at a large distance may be assigned relatively close frequencies.

The use of graphs to model this "channel assignment" problem was first proposed by Hale in 1980 [2]; Chartrand et al introduced the variation known as radio labeling in 2001 [1].

In the graph model of the channel assignment problem, the vertices correspond to the transmitters, and graph distance plays the role of geographical distance. We assume all graphs are connected and simple. The *distance* between two vertices u and v of a graph G, $d(u, v)$, is the length of the shortest path between u and v. The *diameter* of G, $\text{diam}(G)$, is the maximum distance, taken over all pairs of vertices of G. A *radio labeling* of a graph G is then defined to be a function $c : V(G) \to \mathbf{N}$ satisfying

$$d(u, v) + |c(u) - c(v)| \geq 1 + \text{diam}(G)$$

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for all distinct pairs of vertices $u, v \in V(G)$. The *span* of a radio labeling c is the maximum integer assigned by c. The *radio number* of a graph G, $rn(G)$, is the minumum span, taken over all radio labelings of G[1].

We focus on Cartesian products of cycles. We remind the reader that the cycle graph of order n, C_n, may be represented with vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. The diameter of C_n is $\lfloor \frac{n}{2} \rfloor$.

The Cartesian product of two graphs G and H has vertex set $V(G\Box H) = V(G) \times V(H) = \{(g,h) \,|\, g \in V(G) \text{ and } h \in V(H)\}$. The edges of $G\Box H$ consist of those pairs of vertices $\{(g,h), (g',h')\}$ satisfying $g = g'$ and h is adjacent to h' in H or $h = h'$ and g is adjacent to g' in G. We note that $C_n\Box C_n$ has n^2 vertices, and $\text{diam}(C_n\Box C_n) = 2\lfloor \frac{n}{2} \rfloor$.

Section 2 of this paper contains results providing the lower bound for $rn(C_n\Box C_n)$. In Section 3, we define a radio labeling of $C_n\Box C_n$; the span of this labeling is equal to the lower bound, thus establishing the radio number of $C_n\Box C_n$. A comparison of $rn(C_n\Box C_n)$ to $rn(C_n)$ (determined by Liu and Zhu, [3]) is provided in Section 4, wherein is raised a question about the relationship of $rn(G\Box H)$ to $rn(G)$ and $rn(H)$.

2. Lower Bound

The lower bound for $rn(C_n\Box C_n)$ is reached in three steps. First we examine the maximum possible sum of the pairwise distances between any three vertices of C_n. We use this maximum sum to establish a minimum possible "gap" between the i^{th} and $(i+2)^{nd}$ largest labels. Using 1 for the smallest label and taking the size of the gap into account then provides a lower bound for the span of any labeling.

We provide the details of this approach for $C_{2k}\Box C_{2k}$ in Lemmas 2.1 and 2.2 and Theorem 2.3. As the logic of the proofs of the corresponding results for $C_{2k+1}\Box C_{2k+1}$ is identical, we leave the details of Lemma 2.4, Lemma 2.5, and Theorem 2.6 to the reader.

Lemma 2.1. *Let* $u, v, w \in V(C_{2k}\Box C_{2k})$. *Then* $d(u,v) + d(v,w) + d(u,w) \leq 2\,\mathit{diam}(C_{2k}\Box C_{2k})$.

Proof. Express u, v, and w via their component vertices, i.e. as $u = (x_1, y_1)$, $v = (x_2, y_2)$, and $w = (x_3, y_3)$, where x_1, x_2, and x_3 are all vertices of C_{2k}. Then

$$\begin{aligned} d(u,v)+&d(v,w) + d(u,w) \\ &= d\big((x_1,y_1),(x_2,y_2)\big) + d\big((x_2,y_2),(x_3,y_3)\big) + d\big((x_1,y_1),(x_3,y_3)\big) \\ &= d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) + d(y_1,y_2) + d(y_2,y_3) + d(y_1,y_3). \end{aligned}$$

In taking shortest paths between x_1, x_2, and x_3 (all in C_{2k}), one never need take more steps than those necessary to completely traverse C_{2k}, i.e.

$$d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) \leq 2k.$$

The same is true of the sum of the pairwise distances between vertices y_1, y_2, and y_3. Thus

$$d(x_1,x_2) + d(x_2,x_3) + d(x_1,x_3) + d(y_1,y_2) + d(y_2,y_3) + d(y_1,y_3) \leq 4k = 2\,\text{diam}(G).$$

[1]We use the convention that $\mathbf{N}$ consists of the positive integers. Some authors let $\mathbf{N}$ include 0, with the result that radio numbers using this definition are one less than radio numbers determined using the positive integers.

□

We use this maximum possible sum of the pairwise distances between three vertices of $C_{2k}\Box C_{2k}$ together with the radio condition to determine the minimum distance between every other label (arranged in increasing order) in a radio labeling of $C_{2k}\Box C_{2k}$.

Lemma 2.2. *Let c be a radio labeling of $C_{2k}\Box C_{2k}$. Then for any three vertices $u, v, w \in V(C_{2k}\Box C_{2k})$ satisfying $c(w) < c(v) < c(u)$, we have $c(u) - c(w) \geq k + 2$.*

Proof. Since $c(u)$, $c(v)$ and $c(w)$ are radio labels,

$$\begin{aligned} d(u,v) + |c(u) - c(v)| &\geq 1 + \operatorname{diam}(C_{2k}\Box C_{2k}), \\ d(v,w) + |c(v) - c(w)| &\geq 1 + \operatorname{diam}(C_{2k}\Box C_{2k}), \text{ and} \\ d(u,w) + |c(u) - c(w)| &\geq 1 + \operatorname{diam}(C_{2k}\Box C_{2k}). \end{aligned}$$

Summing these inequalities yields

$$d(u,v)+d(v,w)+d(u,w)+c(u)-c(v)+c(v)-c(w)+c(u)-c(w) \geq 3+3\operatorname{diam}(C_{2k}\Box C_{2k}).$$

Furthermore, by Lemma 2.1, $d(u,w) + d(v,w) + d(u,w) \leq 2\operatorname{diam}(C_{2k}\Box C_{2k})$, so we have

$$2\operatorname{diam}(C_{2k}\Box C_{2k}) + 2c(u) - 2c(w) \geq 3 + 3\operatorname{diam}(C_{2k}\Box C_{2k}).$$

As $\operatorname{diam}(C_{2k}\Box C_{2k}) = 2k$, it follows that

$$\begin{aligned} 2(2k) + 2c(u) - 2c(w) &\geq 3 + 3(2k) \\ c(u) - c(w) &\geq \frac{3+2k}{2} = \frac{3}{2} + k. \end{aligned}$$

As $c(u) - c(w)$ is an integer, we may conclude that $c(u) - c(w) \geq 2 + k$. □

Knowledge of the size of the minimum gap allowable between the values of every other label makes it possible to calculate the minimum possible span of a radio labeling of $C_{2k}\Box C_{2k}$.

Theorem 2.3. $rn(C_{2k}\Box C_{2k}) \geq 2k^3 + 4k^2 - k$.

Proof. Let c be a radio labeling of $C_{2k}\Box C_{2k}$. Rename the vertices of $C_{2k}\Box C_{2k}$ using the set $\{x_1, x_2, \ldots, x_{(2k)^2}\}$ so that $c(x_i) < c(x_j)$ whenever $i < j$. Consider the lowest possible values of $c(x_i)$ for each i. We have $c(x_1) \geq 1$ and $c(x_2) \geq 2$. From Lemma 2.2 we know $c(x_3) \geq c(x_1) + k + 2$, and in general,

$$c(x_i) \geq \begin{cases} 1 + \frac{i-1}{2}(k+2), & \text{when } i \text{ is odd} \\ 2 + \frac{i-2}{2}(k+2), & \text{when } i \text{ is even.} \end{cases}$$

Thus $rn(C_{2k}\Box C_{2k}) \geq \operatorname{span}(c) = c(x_{(2k)^2}) \geq 2 + \frac{(2k)^2-2}{2}(k+2) = 2k^3 + 4k^2 - k$. □

A lower bound for the radio number of $C_{2k+1}\Box C_{2k+1}$ may be obtained in much the same way as the lower bound for $C_{2k}\Box C_{2k}$.

Lemma 2.4. *Let $u, v, w \in V(C_{2k+1}\Box C_{2k+1})$. Then*

$$d(u,v) + d(v,w) + d(u,w) \leq 2\,diam(C_{2k+1}\Box C_{2k+1}) + 2.$$

Proof. As in the proof of Lemma 2.1, we write the vertices from $C_{2k+1}\Box C_{2k+1}$ via their components: $u = (x_1, y_1)$, $v = (x_2, y_2)$, and $w = (x_3, y_3)$. Here, however, the sum $d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3)$ may be as much as $2k + 1$ (i.e. once around the cycle). So

$$\begin{aligned}
d(u, v)+&d(v, w) + d(u, w)\\
&= d\big((x_1, y_1), (x_2, y_2)\big) + d\big((x_2, y_2), (x_3, y_3)\big) + d\big((x_1, y_1), (x_3, y_3)\big)\\
&= d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3) + d(y_1, y_2) + d(y_2, y_3) + d(y_1, y_3)\\
&\le 2(2k + 1)\\
&= 2\,\mathrm{diam}(C_{2k+1}\Box C_{2k+1}) + 2.
\end{aligned}$$

□

Lemma 2.5. *Let c be a radio labeling of $C_{2k+1}\Box C_{2k+1}$. Then for any three vertices $u, v, w \in V(C_{2k+1}\Box C_{2k+1})$ satisfying $c(w) < c(v) < c(u)$, we have $c(u) - c(w) \ge k + 1$.*

The proof of Lemma 2.5 is analogous to that of Lemma 2.2: the only change is the substitution of $2\,\mathrm{diam}(C_{2k+1}\Box C_{2k+1}) + 2 = 4k + 2$ for $2\,\mathrm{diam}(C_{2k}\Box C_{2k}) = 4k$.

Theorem 2.6. $rn(C_{2k+1}\Box C_{2k+1}) \ge 2k^3 + 4k^2 + 2k + 1$.

Proof. The proof is analogous to that of Theorem 2.3, with the substitution of $k+1$ (from Lemma 2.5) for $k + 2$ (from Lemma 2.2). Now, for any radio labeling c of $C_{2k+1}\Box C_{2k+1}$, we have

$$c(x_i) \ge \begin{cases} 1 + \frac{i-1}{2}(k + 1), & \text{when } i \text{ is odd}\\ 2 + \frac{i-2}{2}(k + 1), & \text{when } i \text{ is even.}\end{cases}$$

As $C_{2k+1}\Box C_{2k+1}$ has $(2k+1)^2$ vertices, we conclude $rn(C_{2k+1}\Box C_{2k+1}) \ge \mathrm{span}(c) = c(x_{(2k+1)^2}) \ge 1 + \frac{(2k+1)^2-1}{2}(k + 1) = 2k^3 + 4k^2 + 2k + 1$. □

3. Upper Bound

Our general approach to establishing the upper bound for $rn(C_n\Box C_n)$ consists of three steps. We use the convention that $V(C_n\Box C_n) = \{(v, w) \mid 0 \le v, w \le n-1\}$. First we define a position function $p : \{0, 1, \ldots, n^2 - 1\} \to V(C_n\Box C_n)$ and argue that p is a bijection. Defining $x_i = p(i)$ allows us to rename the vertices of $C_n\Box C_n$ in what will be a useful way. Next we give a labeling $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \mathbf{N}$ for which $c(x_0) < c(x_1) < \cdots < c(x_{n^2-1})$. We then prove that c is a radio labeling of $C_n\Box C_n$. (The fact that $c(x_i) < c(x_j)$ when $i < j$ simplifies the proof that c is a radio labeling.) It follows that $rn(C_n\Box C_n) \le \mathrm{span}(c)$.

Recall that any radio labeling c of G must satisfy the radio condition

$$d(u, v) + |c(u) - c(v)| \ge 1 + \mathrm{diam}(G)$$

for all distinct vertices $u, v \in V(G)$. Once $|c(u) - c(v)| \ge \mathrm{diam}(G)$, the radio condition is satisfied for u, v and for any pair of vertices with label difference at least as big as $|c(u) - c(v)|$. The next remark states this fact precisely, and will be of use in limiting the number of vertex pairs for which it must be checked that specific labelings satisfy the radio condition.

Remark 3.1. *Let $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \boldsymbol{N}$ be a labeling of $G = (V, E)$ satisfying $c(x_0) < c(x_1) < \cdots < c(x_{n^2-1})$. If $c(x_{i+j}) - c(x_i) \geq diam(G)$ for some j and for all i, $0 \leq i \leq n - j$, then c satisfies the radio condition for all pairs of vertices x_i, x_{i+k} for all $k \geq j$ and for all i, $0 \leq i \leq n - k$.*

Should the difference in two vertices' labels not exceed the diameter of the graph, we will need to calculate the distance between the vertices. We indicate how to calculate distances below.

Remark 3.2. *Let $(v_i, w_i), (v_j, w_j) \in V(C_n \square C_n)$ (where v_i, w_i, v_j, and w_j are taken mod n). Then*

$$d\big((v_i, w_i), (v_j, w_j)\big) = \min\{|v_i - v_j|, n - |v_i - v_j|\} + \min\{|w_i - w_j|, n - |w_i - w_j|\}.$$

In Section 2 we established a lower bound for $rn(C_n \square C_n)$; this bound depends on the parity of n. The upper bound also depends on the parity of n; the next two theorems establish this upper bound.

Theorem 3.3. *Suppose $n = 2k + 1$. Then $rn(C_n \square C_n) \leq 2k^3 + 4k^2 + 2k + 1$.*

Proof. For each of k even and k odd we provide a radio labeling with span $2k^3 + 4k^2 + 2k + 1$.
Case 1: k is odd.
Define $p : \{0, 1, \ldots, n^2 - 1\} \to \{(v, w) \,|\, 0 \leq v, w \leq 2k\}$ by

$$p(i) = \left(ik, r + i\left(\frac{k+1}{2}\right)\right) \bmod n, \text{ where } r = \left\lfloor \frac{i}{n} \right\rfloor.$$

We wish to show that p is a bijection. Suppose that $p(i) - p(j) = 0$ for some $i \neq j$. Examining the first component of $p(i) - p(j)$, we see that $ik - jk \equiv 0 \bmod n$. As k and n are relatively prime, $i - j \equiv 0 \bmod n$. The second component of $p(i) - p(j)$ must also be zero: this gives

$$\begin{aligned} 0 &\equiv \left(\left\lfloor \frac{i}{n} \right\rfloor + i\left(\frac{k+1}{2}\right)\right) - \left(\left\lfloor \frac{j}{n} \right\rfloor + j\left(\frac{k+1}{2}\right)\right) \bmod n \\ &= \left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor + (i - j)\left(\frac{k+1}{2}\right) \bmod n \\ &= \left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor \bmod n. \end{aligned}$$

But $i \neq j$ and $i \equiv j \bmod n$ imply $\lfloor \frac{i}{n} \rfloor - \lfloor \frac{j}{n} \rfloor \not\equiv 0 \bmod n$. Thus $p(i) \neq p(j)$ for distinct i, j in the domain of p, and we may conclude that p is a bijection.

We now use the elements of the set $\{x_0, x_1, \ldots, x_{n^2-1}\}$ to rename the vertices of $C_n \square C_n$ by agreeing that $p(i) = x_i$. Define the labeling $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \mathbf{N}$ by

$$c(x_i) = 1 + i\left(\frac{k+1}{2}\right).$$

Claim: The labeling c is a radio labeling of $rn(C_n \square C_n)$.
To establish our claim we must show that c satisfies the radio condition

$$d(u, v) + |c(u) - c(v)| \geq \text{diam}(G) + 1 = 2k + 1$$

for all $u, v \in V(C_n \square C_n)$. Note that $c(x_{i+4}) - c(x_i) \geq \text{diam}(G) = 2k$ for all $i = 0, \ldots n^2 - 4$, so Remark 3.1 indicates that we need only verify that c satisfies the radio condition for vertex pairs x_i, x_{i+j} with $j \leq 3$.

We will examine first pairs of vertices with fixed r, i.e. with indices in $\{an, an+1, \ldots, an+n-1\}$ for $a = 0, 1, \ldots n-1$. Subsequently we will show that the radio condition is satisfied for vertices of the form x_i, x_{i+j} where $\lfloor \frac{i}{n} \rfloor \neq \lfloor \frac{i+j}{n} \rfloor$ and $j \leq 3$.

Subcase 1: Take $x_i, x_{i+j} \in \{an, an+1, \ldots, an+n-1\}$ for $a = 0, 1, \ldots n-1$ and $j \leq 3$. Let $n \geq 5$. The distance between x_i and x_{i+j} is given by examining the position function p and using Remark 3.2.

vertex pair	$d(x_i, x_{i+j})$	$c(x_{i+j}) - c(x_i)$	$d(x_i, x_{i+j}) + \lvert c(x_j) - c(x_i)\rvert$
x_i, x_{i+1}	$k + \frac{k+1}{2}$	$\frac{k+1}{2}$	$2k+1$
x_i, x_{i+2}	$1 + k$	$k+1$	$2k+2$
x_i, x_{i+3}	$k - 1 + \frac{k-1}{2}$	$\frac{3k+3}{2}$	$3k$

Each sum in the last column is at least $2k+1$ (recalling $n \geq 5$), so this completes the argument that the radio condition is satisfied by c for all $x_i, x_j \in V(C_n \square C_n)$ when $n \geq 5$.

Subcase 2: Consider x_i, x_{i+j} with $j \leq 3$ and $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i+j}{n} \rfloor - 1$. Compare the calculation of $d(x_i, x_{i+j})$ here with the analogous calculation in Subcase 1. The new condition introduced here, that $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{j}{n} \rfloor - 1$, may change the distance by ± 1 (as r changes by 1 in the second component of $p(i+j)$). The previous verification that the radio condition holds thus suffices here for (x_i, x_{i+2}) and (x_i, x_{i+3}), as the sum of the distance and the label difference exceeded $2k+1$. We recalculate $d(x_i, x_{i+1})$:

$$\begin{aligned}
d(x_i, x_{i+1}) &= d(x_{an-1}, x_{an}) \\
&= d\left(\left((an-1)k, (a-1) + (an-1)\frac{k+1}{2}\right), \left(ank, a + (an)\frac{k+1}{2}\right)\right) \\
&\equiv d\left(\left(-k, a - \frac{k+3}{2}\right), (0, a)\right) \\
&= k + \frac{k+3}{2}.
\end{aligned}$$

The distance increases; the radio condition is met.

The two subcases show that c is a radio labeling of $C_n \square C_n$ when $n \geq 5$. We specified $n \geq 5$ to take advantage of Remark 3.1, however, c is also a radio labeling of $C_3 \square C_3$. To see this, let $n = 3$ and note that $p(i)$ and $p(i+1)$ differ in both components for all $i = 0, 1, \ldots, 8$. Thus $d(x_i, x_{i+1}) \geq 2$; as $\mathrm{diam}(C_3 \square C_3) = 2$, we have $d(x_i, x_{i+1}) = 2$ for $i = 0, 1, \ldots, 8$. When $|j-i| \geq 2$, we have $|c(x_j) - c(x_i)| \geq 2$. These two facts ensure that the radio condition is satisfied by all pairs of vertices of $C_3 \square C_3$.

This establishes the claim that c is a radio labeling of $C_n \square C_n$ (when $n = 2k+1$ and k is odd). To calculate the span of c, we use the fact that c is an increasing function to note that

$$\begin{aligned}
span(c) = c(x_{n^2-1}) &= 1 + (n^2 - 1)\left(\frac{k+1}{2}\right) \\
&= 1 + \left((2k+1)^2 - 1\right)\left(\frac{k+1}{2}\right) \\
&= 2k^3 + 4k^2 + 2k + 1.
\end{aligned}$$

Case 2: k is even.
As $C_1 \square C_1$ has only one vertex, we label this vertex 1; the result follows.

So assume $k \geq 2$ ($n \geq 5$). As in Case 1, we let $r = \lfloor \frac{i}{n} \rfloor$. Define $p : \{0, 1, \ldots, n^2 - 1\} \to \{(v, w) \,|\, 0 \leq v, w \leq 2k\}$ by

$$p(i) = \begin{cases} \left(\frac{i}{2}(k+1),\ r + ik\right) \mod n, & \text{when } i \text{ is even,} \\ \left(\frac{i-1}{2}(k+1) + \frac{k}{2} + 1,\ r + ik\right) \mod n, & \text{when } i \text{ is odd,} \end{cases}$$

Again we assume that $p(i) - p(j) = 0$ for $i \neq j$. When $i \equiv j \mod n$, the second component of $p(i) - p(j)$ is not zero. So assume $i, j \in \{an, an+1, \ldots, an+n-1\}$ for some $a \in \{0, 1, \ldots n-1\}$. Reexamine the second component of $p(i) - p(j)$: $\left(\lfloor \frac{i}{n} \rfloor + ik\right) - \left(\lfloor \frac{j}{n} \rfloor + jk\right) \equiv (i-j)k \mod n \equiv 0 \mod n$ if and only if $i = j$ (as k and n are relatively prime). Thus $p(i) - p(j) = 0$ exactly when $i = j$, so p is a bijection.

We again rename the vertices of $C_n \square C_n$ using the set $\{x_0, x_1, \ldots, x_{n^2-1}\}$ by specifying $p(i) = x_i$. The labeling in this case is given by $c : \{x_0, x_1, \ldots, x_{n^2-1}\} \to \mathbf{N}$, where

$$c(x_i) = \begin{cases} 1 + \frac{i}{2}(k+1), & \text{when } i \text{ is even,} \\ 1 + \frac{i-1}{2}(k+1) + \frac{k}{2}, & \text{when } i \text{ is odd.} \end{cases}$$

As $c(x_{i+4}) - c(x_i) \geq \operatorname{diam}(C_n \square C_n)$ for all $i = 0, 1, \ldots, n^2 - 4$, we again apply Remark 3.1 to limit the vertex pairs for which we must verify that c is a radio labeling. We show first that all vertex pairs of the form x_{a+i}, x_{a+i+j} with $a = 0, 1, \ldots, n-1$, $i = 0, 1, \ldots, n-4$ and $j = 1, 2, 3$ satisfy the radio condition:

vertex pair	$d(x_i, x_{i+j})$	$c(x_{i+j}) - c(x_i)$	$d(x_i, x_{i+j}) + \lvert c(x_j) - c(x_i) \rvert$
$x_i,\ x_{i+1}$ (i even)	$\frac{3k}{2} + 1$	$\frac{k}{2}$	$2k + 1$
$x_i,\ x_{i+1}$ (i odd)	$\frac{3k}{2}$	$\frac{k}{2} + 1$	$2k + 1$
$x_i,\ x_{i+2}$	$k + 1$	$k + 1$	$2k + 2$
$x_i,\ x_{i+3}$ (i even)	$\frac{3k}{2} - 2$	$\frac{3k}{2} + 1$	$3k - 1$
$x_i,\ x_{i+3}$ (i odd)	$\frac{3k}{2} - 1$	$\frac{3k}{2} + 2$	$3k + 1$

As $k \geq 2$, all sums in the last column are at least $\operatorname{diam}(C_{2k+1} \square C_{2k+1}) + 1 = 2k + 1$.

As in Subcase 2 of Case 1, we now consider the vertex pairs of the form x_i, x_{i+j} where $j \leq 3$ and $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i+j}{n} \rfloor - 1$. Again the value of $d(x_i, x_{i+j})$ in the table above may change by at most 1 with the introduction of the condition $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i+j}{n} \rfloor - 1$. Consequently, we must verify that the new distance is not less than the previous for (x_i, x_{i+1}) and for (x_i, x_{i+3}) when i is even and $k = 2$. Examine $d(x_i, x_{i+1})$: to fulfill $\lfloor \frac{i}{n} \rfloor = \lfloor \frac{i+1}{n} \rfloor - 1$ we have $i = an - 1$. When i is even, we have

$$\begin{aligned} &d(x_i, x_{i+1}) = d(x_{an-1}, x_{an}) \\ &\quad = d\left(\left(\frac{an-1}{2}(k+1), (a-1) + (an-1)k\right), \left(\frac{an-1}{2}(k+1) + \frac{k}{2} + 1, a + (an)k\right)\right) \\ &\quad = \left(\frac{k}{2} + 1\right) + k = \frac{3k}{2} + 1, \end{aligned}$$

i.e. there is no change in the distance. When $i = an - 1$ is odd, we calculate

$$\begin{aligned} &d(x_i, x_{i+1}) = d(x_{an-1}, x_{an}) \\ &\quad = d\left(\left(\frac{an-2}{2}(k+1) + \frac{k}{2} + 1, (a-1) + (an-1)k\right), \left(\frac{an}{2}(k+1), a + (an)k\right)\right) \\ &\quad = \frac{k}{2} + k = \frac{3k}{2}, \end{aligned}$$

as before.

Now consider (x_i, x_{i+3}) for i is even and $k = 2$. Observe that assuming i is even implies that any of (x_{an-3}, x_{an}), (x_{an-2}, x_{an+1}), or (x_{an-2}, x_{an+2}) may be used to assess the change in distance of (x_i, x_{i+3}). So

$$\begin{aligned}
&d(x_i, x_{i+3}) = d(x_{an-3}, x_{an})\\
&\quad = d\left(\left(\frac{an-3}{2}(k+1), (a-1) + (an-3)k\right), \left(\frac{an-1}{2}(k+1) + \frac{k}{2} + 1, a + (an)k\right)\right)\\
&\quad = \frac{k}{2} - 1 + k = \frac{3k}{2} - 1,
\end{aligned}$$

as before. Again, the radio condition is satisfied in all cases.

To show that the radio number of $C_{2k+1}\Box C_{2k+1}$ is as claimed, we compute the span of this labeling:

$$\begin{aligned}
span(c) = c(x_{n^2-1}) &= 1 + \frac{n^2-1}{2}(k+1)\\
&= 1 + \frac{(2k+1)^2-1}{2}(k+1)\\
&= 2k^3 + 4k^2 + 2k + 1.
\end{aligned}$$

□

The proof of Theorem 3.4 has a similar structure to that of Theorem 3.3, but the position and labeling functions are independent of the parity of k.

Theorem 3.4. *Let $n = 2k$. Then $rn(C_n\Box C_n) \leq 2k^3 + 4k^2 - k$.*

Proof. Define $p : \{0, 1, \ldots, n^2-1\} \to \{(v, w) \,|\, 0 \leq v, w \leq 2k-1\}$ by

$$p(i) = \begin{cases} (r, kr+s) \bmod n, & \text{when } i \equiv 0 \bmod 4,\\ (r+k, kr+s+k) \bmod n, & \text{when } i \equiv 1 \bmod 4,\\ (r, kr+s+k) \bmod n, & \text{when } i \equiv 2 \bmod 4,\\ (r+k, kr+s) \bmod n, & \text{when } i \equiv 3 \bmod 4, \end{cases}$$

$$\text{where } r = \left\lfloor \frac{i}{2n} \right\rfloor \text{ and } s = \left\lfloor \frac{i}{4} \right\rfloor \bmod k.$$

Claim: p is a bijection. Consider the following possibilities for the relationship of the indices i and j:

(1) $i \not\equiv j \bmod 4$; i and j have opposite parity,
(2) $i \not\equiv j \bmod 4$; i and j have the same parity,
(3) $i \equiv j \bmod 4$; $\lfloor \frac{i}{2n} \rfloor \neq \lfloor \frac{j}{2n} \rfloor$, and
(4) $i \equiv j \bmod 4$; $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$.

In the first case, the first components of $p(i)$ and $p(j)$ agree exactly when $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor + k$. But $\lfloor \frac{i}{2n} \rfloor \leq \left\lfloor \frac{n^2-1}{2n} \right\rfloor = \frac{n-2}{2} < k$, so this is impossible.

Suppose $i \not\equiv j \bmod 4$ and i and j have the same parity. If $\lfloor \frac{i}{2n} \rfloor \neq \lfloor \frac{j}{2n} \rfloor$ then the first component of $p(i) - p(j)$ is not zero. Should $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$, examine the second component:

$$\begin{aligned} p(i) - p(j) &= \left| \left(k \left\lfloor \frac{i}{2n} \right\rfloor + \left\lfloor \frac{i}{4} \right\rfloor \mod k \right) - \left(k \left\lfloor \frac{j}{2n} \right\rfloor + \left\lfloor \frac{j}{4} \right\rfloor \mod k + k \right) \right| \\ &\equiv \left| \left(\left\lfloor \frac{i}{4} \right\rfloor - \left\lfloor \frac{j}{4} \right\rfloor \mod k \right) - k \right| \mod 2k \\ &\neq 0. \end{aligned}$$

In the third case, the first components of $p(i)$ and $p(j)$ will be the same only when they are equivalent $\mod n$. But the possible values for $r = \lfloor \frac{i}{2n} \rfloor$ never reach n, so we may rule out this eventuality. Finally, in the fourth case, note that the hypotheses imply that $p(i) = p(j)$ if and only if $s_i = \lfloor \frac{i}{4} \rfloor \mod k = s_j = \lfloor \frac{j}{4} \rfloor \mod k$. But $|i-j| < 2n = 4k$, so $\left| \lfloor \frac{i}{4} \rfloor - \lfloor \frac{j}{4} \rfloor \right| < k$, thus $s_1 \neq s_2$. Therefore we may conclude that p is a bijection.

Again, we rename the vertices of $C_n \square C_n$ by agreeing that $p(i) = x_i$. The labeling is given by $c : \{x_0, x_1, \dots, x_{n^2-1}\} \to \mathbf{N}$ by

$$c(x_i) = \begin{cases} 1 + \frac{i}{2}(k+2), & \text{when } i \text{ is even,} \\ 2 + \frac{i-1}{2}(k+2), & \text{when } i \text{ is odd.} \end{cases}$$

Subcase 1: As $c(x_{i+4}) - c(x_i) \geq 2k = \operatorname{diam}(C_n \square C_n)$ for all $i = 0, 1, \dots, n^2 - 4$, we again apply Remark 3.1 to limit the vertex pairs for which we must verify that c is a radio labeling. Consider first pairs of vertices $\{x_i, x_j\}$ with $|i - j| \leq 3$ and $\lfloor \frac{i}{2n} \rfloor = \lfloor \frac{j}{2n} \rfloor$. For $i_2 = i_1 + 4m$ and $j_2 = j_2 + 4m$. where m is an integer, we have $d(x_{i_1}, x_{j_1}) = d(x_{i_2}, x_{j_2})$ and $|c(x_{i_1}) - c(x_{j_1})| = |c(x_{i_2}) - c(x_{j_2})|$, so consideration of the pairs for which the distances and the label differences are shown in the tables below suffices. The first table gives the label differences.

	x_1	x_2	x_3	x_4	x_5
x_0	1	$k+2$	$k+3$		
x_1		$k+1$	$k+2$	$2k+3$	
x_2			1	$k+2$	$k+3$
x_3				$k+1$	$k+2$

The second table gives the distances between vertices.

	x_1	x_2	x_3	x_4	x_5
x_0	$2k$	k	k		
x_1		k	k	$2k-1$	
x_2			$2k$	$k-1$	$k+1$
x_3				$k+1$	$k-1$

Summing the corresponding entries from each table shows that the radio condition is satisfied in all cases.

Subcase 2: It remains only to verify that the radio condition holds for vertices with index differences less than four and indices near a multiple of $2n$. Specifically, we must calculate $d(u, v) + |c(u) - c(v)|$ for all vertices $\{u, v\}$ of the form $\{x_{an-3}, x_{an}\}$, $\{x_{an-2}, x_{an}\}$, $\{x_{an-2}, x_{an+1}\}$, $\{x_{an-1}, x_{an}\}$, $\{x_{an-1}, x_{an+1}\}$, and $\{x_{an-1}, x_{an+2}\}$, where a is an even integer. Note that a is even implies that

$an = a2k \equiv 0 \bmod 4$. Also, as $n \geq 2$, we know that

$$r = \left\lfloor \frac{i}{2n} \right\rfloor = \begin{cases} \left\lfloor \frac{an-t}{2n} \right\rfloor = \frac{a}{2} - 1, & \text{for } i = an - t \text{ and } t = 1,2,3 \\ \left\lfloor \frac{an+t}{2n} \right\rfloor = \frac{a}{2}, & \text{for } i = an + t \text{ and } t = 0,1,2. \end{cases}$$

and

$$s = \left\lfloor \frac{i}{4} \bmod k \right\rfloor = \begin{cases} \frac{ak}{2} - 1 \bmod k = k - 1, & \text{for } i = an - t \text{ and } t = 1,2,3 \\ \frac{ak}{2} \bmod k = 0, & \text{for } i = an + t \text{ and } t = 0,1,2. \end{cases}$$

Accordingly, we calculate the position functions (each considered mod n) and the label values for each vertex of interest.

$$\begin{aligned}
x_{an-3} &= \left(\frac{a}{2} - 1 + k, k\left(\frac{a}{2} - 1\right) - 1\right); & c(x_{an-3}) &= 2 + \frac{an-4}{2}(k+2) \\
x_{an-2} &= \left(\frac{a}{2} - 1, k\left(\frac{a}{2} - 1\right) - 1\right); & c(x_{an-2}) &= 1 + \frac{an-2}{2}(k+2) \\
x_{an-1} &= \left(\frac{a}{2} - 1 + k, k\left(\frac{a}{2} - 1\right) + (k-1)\right); & c(x_{an-1}) &= 2 + \frac{an-2}{2}(k+2) \\
x_{an} &= \left(\frac{a}{2}, k\frac{a}{2}\right); & c(x_{an}) &= 1 + \frac{an}{2}(k+2) \\
x_{an+1} &= \left(\frac{a}{2} + k, k\frac{a}{2} + k\right); & c(x_{an+1}) &= 2 + \frac{an}{2}(k+2) \\
x_{an+2} &= \left(\frac{a}{2}, k\frac{a}{2} + k\right); & c(x_{an+2}) &= 1 + \frac{an+2}{2}(k+2)
\end{aligned}$$

This allows the calculation of the distance between vertices and the difference of the labels for each vertex pair in question.

vertex pair u, v	$d(u,v)$	$\lvert c(u) - c(v)\rvert$	$d(u,v) + \lvert c(u) - c(v)\rvert$
x_{an-3}, x_{an}		$2k+3$	$> 2k+3$
x_{an-2}, x_{an}	k	$k+2$	$2k+2$
x_{an-1}, x_{an}	k	$k+3$	$2k+3$
x_{an-2}, x_{an+1}	k	$k+3$	$2k+3$
x_{an-1}, x_{an+1}	k	$k+2$	$2k+2$
x_{an-1}, x_{an+2}		$2k+3$	$> 2k+3$

The radio condition is satisfied in all cases.

Finally, we compute the span of this radio labeling:

$$span(c) = c(x_{n^2-1}) = 2 + \frac{(2k)^2 - 2}{2}(k+2) = 2k^3 + 4k^2 - k.$$

□

As each upper bound for the radio number of the Cartesian product of a cycle with itself is equal to the corresponding lower bound, we have the radio numbers themselves.

Theorem 3.5. *Let k be a nonnegative integer. Then $rn(C_{2k} \square C_{2k}) = 2k^3 + 4k^2 - k$ and $rn(C_{2k+1} \square C_{2k+1}) = 2k^3 + 4k^2 + 2k + 1$.*

4. Additional Comments

It would be interesting to determine the general relationship between the radio number of a Cartesian product of graphs and the radio numbers of the factor graphs (i.e. between $rn(G\square H)$, $rn(G)$, and $rn(H)$). For instance, $rn(K_m) = m$ and $rn(K_m\square K_n) = mn = |V(K_m\square K_n)|$. While this might lead to the hope that $rn(G\square H) = rn(G)rn(H)$, the result in this paper together with Liu and Zhu's result on the radio number of cycles [3] shows that this is not the case:

$$rn(C_{2k}) = k^2 + k + 2, \text{ and}$$

$$rn(C_{2k+1}) = \begin{cases} k^2 + k + 1, & \text{when } k \text{ is even,} \\ k^2 + 2k + 1, & \text{when } k \text{ is odd,} \end{cases}$$

whereas

$$rn(C_{2k}\square C_{2k}) = 2k^3 + 4k^2 - k, \text{ and}$$

$$rn(C_{2k+1}\square C_{2k+1}) = 2k^3 + 4k^2 + 2k + 1.$$

From this, we see that $rn(C_n\square C_n)$ is markedly less than $rn(C_n)rn(C_n)$. At this point, not enough is known about radio numbers of Cartesian products to venture a conjecture regarding the relationship of $rn(G\square H)$ to $rn(G)$ and $rn(H)$.

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