Math 451 Final Exam  
Due Friday, May 14, 11:59 pm

Instructions: This exam is open book and open note. You may also ask me questions. You should not need any outside references to complete the exam. If you do consult an outside source or person about a problem, please cite the work or individual in your solution. Submit your completed exam via email, either using a type-setting program like \LaTeX{} or scanning a hand-written document.

All steps in solutions to these problems should be justified using techniques and theorems we have learned this semester. Remember to focus, and to save 10% for happiness! Note: the points add up to more than 100, but your score will be out of 100.

1. Compute the following integrals. When using a theorem for the computation, (i) state the theorem, (ii) state the hypotheses of the theorem, and (iii) justify why the integral being computed satisfies those hypotheses. (5 pts each)
   (a) For $\Gamma$ given by the counterclockwise circle of radius 1 around the point $z_0 = 2 + i$, compute $\int_{\Gamma} e^{\sin \frac{z}{z+1}} \, dz$.
   (b) For $\Gamma$ given by the semi-circle from $-3i$ to $3i$, compute $\int_{\Gamma} \frac{1}{z} \, dz$.
   (c) For $\Gamma$ given by the semi-circle from $-3i$ to $3i$, compute $\int_{\Gamma} \frac{1}{z-5-i} \, dz$.
   (d) For $\Gamma$ given by the circle of radius 3 around the point $z_0 = 4i$, compute $\int_{\Gamma} \frac{(z-1)^2 \sin(e^{\cos z^2})}{z^3(z^2+16)} \, dz$.

2. Find all functions $f(z)$ that are analytic on the domain $D = \{ z \in \mathbb{C} \mid |z| < R, R > 0 \}$ and satisfy $f(0) = i$ and $|f(z)| \leq 1$ on $D$. (10 pts)

3. (10 pts each)
   (a) Show that if a function $g(z)$ can be written as $g(z) = \frac{f(z)}{(z-z_0)}$ for $f(z)$ analytic that $g(z)$ has a series representation as $\sum_{i=-1}^{\infty} a_i z^i$ that converges everywhere except $z = z_0$, where its value goes to $\infty$.
   (b) Show that if a function $g(z)$ can be written as $g(z) = \frac{f(z)}{(z-z_0)^m}$ for $f(z)$ analytic that $g(z)$ has a series representation as $\sum_{i=-m}^{\infty} a_i z^i$ that converges everywhere except $z = z_0$, where its value goes to $\infty$.
   (c) Express the coefficients $\{a_i\}$ in terms of the Cauchy Integral Formula.

4. A function $f(z)$ is univalent when it is one-to-one (so $f(z_1) = f(z_2) \iff z_1 = z_2$). Functions that are analytic and univalent are lovely because they have analytic inverses. (10 pts each)
   (a) Show that $f(z) = (1+z)^2$ is univalent in the unit disk $|z| < 1$, but that $g(z) = (1+z)^4$ is not. What is the difference? (Hint: To show $f$ is univalent, plug into the definition of univalent and use algebra. To show $g$ is not univalent, look at what gets mapped by $g$ onto the real axis between 0 and -4. Find a point in that part of the range that came from different points in the domain.)
   (b) Use the Taylor series expansion for a univalent, analytic function $f(z)$ to show that associated to every such $f(z)$ is a $g(z) = z + c_2 z^2 + c_3 z^3 + \ldots$. (Hint: If $f(x)$ is invertible, what does that say about $f'(x)$?)
(c) Show that \( g(z) = z + a_2 z^2 \) is univalent on the unit disk if and only if \(|a_2| \leq 1/2\). (Hint: look at where the boundary of the disk (i.e., the unit circle) goes under \( g \). Write \( z = \cos \theta + i \sin \theta \) and then look at real and imaginary parts of \( g \).)

(d) Finding the \( g(z) \) for univalent, analytic functions creates equivalence classes for them, where \( f_1(z) \sim f_2(z) \) when they are associated to the same \( g(z) \). Characterize the analytic, univalent functions \( f(z) \) associated to \( g(z) = z + a_2 z^2 \) for \(|a_2| \leq 1/2\).

(5) Suppose \( f(z) \) is analytic on a domain \( D \), and the line \( L \) joining two points \( z_1, z_2 \), is entirely contained in \( D \). If \( f'(z) \equiv 0 \) on \( L \), what can you say about \( f(z) \) in \( D \)? Support your claim with a proof. (10 points)