Math 451 Midterm Exam
Due Friday, March 19, 11:59 pm

Instructions: This exam is open book and open note. You may also ask me questions. You should not need any outside references to complete the exam. If you do consult an outside source or person about a problem, please cite the work or individual in your solution. Submit your completed exam via email, either using a type-setting program like \LaTeX or scanning a hand-written document. Refer to the course syllabus for information about academic honesty.

(1) Consider the function \( f(z) = \frac{2z+1}{z+1} \).
   (a) Where is \( f(z) \) analytic?
   (b) Compute a Taylor series expansion for \( f(z) = 0 \). What is the radius of convergence?
   (c) Find \( \lim_{z \to -\infty} f(z) \).
   (d) Where does \( f(z) \) map the real axis?
   (e) Where does \( f(z) \) map the unit circle?

(2) Now consider the general formulation of \( f(z) = \frac{az+b}{cz+d} \). Express your answers in terms of the coefficients \( a, b, c, d \in \mathbb{R} \).
   (a) Where is it analytic?
   (b) Find \( \lim_{z \to -\infty} f(z) \).
   (c) Where does \( f(z) \) map the real axis?
   (d) Where does \( f(z) \) map the unit circle?
   (e) Prove that the composition of any two such functions is of the same form.
   (f) Prove that if \( g(z) \) is the inverse of \( f(z) \) (i.e., \( f \circ g(z) = g \circ f(z) = 1 \)), that \( g(z) \) is again of the same form.
   (g) Identify conditions on the coefficients \( a, b, c, d \) for such a \( g(z) \) to exist.

(3) Suppose \( f(z) \) is an analytic function on a domain \( D \) that is not constant on \( D \). Show that there is no neighborhood \( N \subset D \) where the partial derivatives \( u_x = v_y \equiv 0 \) on \( N \).

(4) Prove that if \( f(z) \) and \( \overline{f(z)} \) are both analytic in a domain \( D \), then \( f(z) = \overline{f(z)} \) is a constant.

(5) Consider the function \( f(z) = z^n, \ n \in \mathbb{R} \). We can analyze what happens to particular points in the complex plane as this function is iterated (applied repeatedly) to those points. Given a point \( z = z_0 \), the orbit of \( z_0 \) under the action of \( f(z) \) is the collection of points that result from the iteration, namely the set \( \{ z_0, f(z_0), f(f(z_0)), f(f(f(z_0))), \ldots \} \). Write \( z_0 = r_0 e^{i \theta_0} \).
   (a) Take \( r_0 = 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be finite?
   (b) Take \( r_0 = 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be infinite?
   (c) Take \( r_0 > 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be finite?
   (d) Take \( r_0 > 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be infinite?
   (e) Take \( r_0 < 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be finite?
   (f) Take \( r_0 < 1 \). For which values of \( n \) will the orbit of \( f(z_0) \) be infinite?
   (g) A point \( z = \xi \) is an attractor if \( f(\xi) = \xi \) and there is a \( z_0 \) whose orbit converges to \( \xi \). For \( n \in \mathbb{N} \), identify possible attractors and find choices for points \( z_0 \) whose orbits converge to the attractors. If no such \( z_0 \) exists for a potential attractor, explain why.