STATEMENT OF RESEARCH

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Introduction: My field of study is **knot theory**. A knot (resp. link) k is a smooth embedding of S^1 (resp. a disjoint union of circles) in $S^3 = \mathbb{R}^3 \cup \{\pm\infty\}$ (e.g. the most basic knots are depicted in Fig. 1).

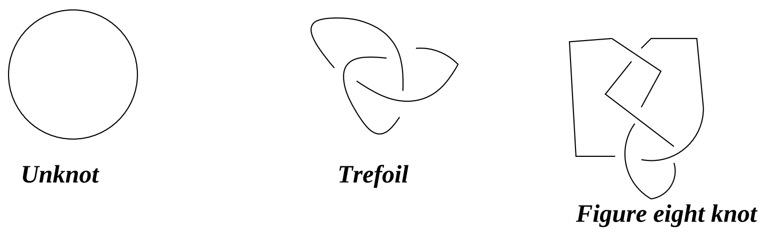

Figure 1:

I am interested in the interplay between knot theory and dimension four topology and geometry, via *twisting operations.* To define a twisting operation, we need to define Dehn surgery.

Dehn surgery: (see Figure 2) Let $N(k)$ be a tubular neighborhood of a knot k the S^3. Then a 3-manifold obtained by $\dfrac{p}{q}$-Dehn surgery ($\dfrac{p}{q} \in \mathbb{Q} \cup \{\pm\infty\}$) along k in S^3 is the new 3-manifold denoted $S^3(k, \dfrac{p}{q}) = (S^3 - intN(k)) \cup N(k)$ such that a meridian of $\partial N(k) \cong T^2$ is identified to a simple closed curve of slope $\dfrac{p}{q}$. For illustration, The Poincaré dodecahedral space is obtained by performing a $\frac{1}{1}$-Dehn surgery along the trefoil knot (see Figure 2).

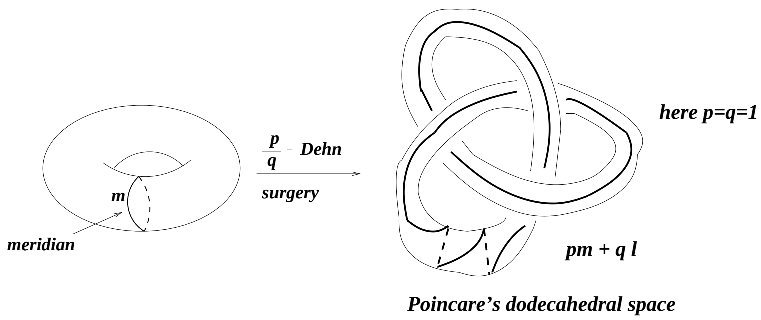

Figure 2:

Twisted knots: (see Figure 3) Let K be a *trivial* knot K in S^3, and a disk D^2 intersecting K in its interior. Let $\omega = \mid \mathrm{lk}(\partial D^2, K) \mid$, and n an integer. A $(-1/n)$-Dehn surgery along ∂D^2 changes K into a new knot K_n in S^3. We say that K_n is obtained from K by (n, ω)*-twisting* (Figure 3 shows that K_{+1} is the trefoil knot). Then we write $K \overset{(n,\omega)}{\rightarrow} K_n$. The disk D is called the *twist disk.*

Research in twisting of knots started in 1990. One pioneer was my Ph.D thesis advisor Y. Mathieu [53] who asked the following questions:

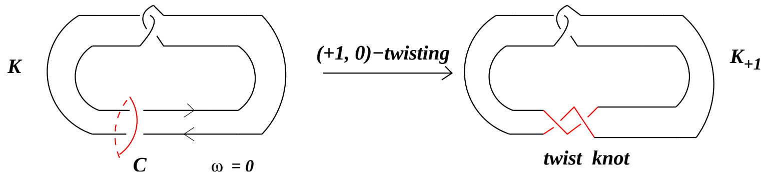

Figure 3:

(Q1) *Can we untie any knot by one twist disk ? and if not*

(Q2) *what is the minimal number of twist disks ?*

(Q3) *Is there a composite twisted knot ?*

These questions have given rise to several research papers. A Japanese team (K. Motegi, C. Hayashi, K. Miyazaki, A. Yasuhara, M. Teragaito , Y. Ohyama, H. Goda) have been interested in twisting operations problems, in addition to J. Song, C. Goodman-Strauss, Daniel Matignon and myself.

In my Ph.D thesis [2], and in a joint work with A. Yasuhara [12], we gave an infinite family of non-twisted torus knots, using some dimension four techniques such as old gauge theory, which answers ($Q1$).

Y. Ohyama [66] showed that *we can untie every knot by two disks,* which answers ($Q2$).

Hayashi-Motegi [64], and M. Teragaito [77] found independently *examples of composite twisted knots,* which answers ($Q3$). In addition, Hayashi-Motegi [64] and C. Goodman-Strauss [31] proved independently that, only single twisting (i.e. $\mid n \mid= 1$) can yield a composite knot.

The techniques I have been using derive from two main branches: **low dimensional Topology** and **Riemannien geometry**. In dimension three topology, I used combinatorial methods (graphs of intersection as in CGLS [17] and Jaco-Shallen-Johannsson decomposition (see [42] and [43])), the well-developed and powerfull tool, customarily used to study Dehn surgery.

There is a connection between twisting operations and dimension four: We can prove, using Kirby Calculus and some twisting manipulations, that a (n, ω)*-twisted knot* in S^3 bounds a properly embedded smooth disk Δ in a punctured standard four manifold of the form $n\mathbb{C}P^2 - intB^4$ if $n < 0$, or $\mid n \mid \overline{\mathbb{C}P^2} - intB^4$ if $n > 0$, or $S^2 \times S^2 - intB^4$ if n is even. The second homolog of $[\Delta]$ can be computed from n and ω.

I was attracted to geometric topology because of this connection, and the richness it acquired from old and new gauge theory: There are several restrictions on the homology classes in a 4-manifold represented by embedded surfaces of given genus (V.A. Rochlin, J. Milnor, R. Kirby, Gilmer-Viro, K. Kikuchi, R.A. Litherland, Seiberg-Witten, Oszvath-Szabo, Fintushel-Stern, Kronheimer-Mrowka, Wintraub, Yammamoto..et al). I am interested in investigating such connections, and two main applications, so far, are:

- Slice genus of (torus) knots in $\mathbb{C}P^2$ (see [3]) (section 5 of this statement).
- Counterexamples to Terry Lawson's conjectures ([7], [3]) (Conjecture 5.1. and 6.1. in page 6).

RESEARCH DONE

By Thurston's uniformization theorem [79] and Jaco-Shallen-Johannsson torus theorem ([42] and [43]), every knot in S^3 is either a torus knot, or a satellite knot, or a hyperbolic knot. In a joint work with A. Yasuhara [12], we studied *twisting of torus knots*. In parallel, in a joint work with D. Matignon and K. Motegi [10], we studied *twisting of graph knots* [10], *twisting of satellite knots* [13] and the *geometric structure of twisted knots* as well as the *Gromov invariant of twisted knots.* I also investigate the smooth slice genus of knots in $\mathbb{C}P^2$, the minimal genus problem, and introduced a new invariant of knots.

1. Twisting of torus knots [12]

A (p,q)*-torus knot* $T(p,q)$ is a knot that wraps around the standard solid torus in the longitudinal direction p times and the meridional direction q times. Note that p and q are coprime. A torus knot $T(p,q)$ $(0 < p < q)$ is *exceptional* if $q \equiv \pm 1 \pmod p$, and *non-exceptional* if it is not exceptional. Let $p(\geq 2)$ be an integer. It is easily seen that $T(p,\pm 1) \overset{(k,p)}{\to} T(p, kp \pm 1)$. Since $T(p,\pm 1)$ is a trivial knot, $T(p, kp \pm 1)$ belongs to $\mathcal{T}$, where $\mathcal{T}$ denote the set of knots that are obtained from a trivial knot by a single twisting. Since the knots $T(2,q)$, $T(3,q)$, $T(4,q)$ and $T(6,q)$ are exceptional, then they belong to $\mathcal{T}$. So we are faced with the following problem:

Problem 1.1. *Is there a torus knot that is not contained in* $\mathcal{T}$*?*

To answer this question, we prove the following:

Proposition 1.1. $T(5,8)$ does not belong to $\mathcal{T}$.

We even give an infinite family as follows:

Theorem 1.1. *Let p be an odd integer. If $p \geq 9$, then $T(p, p+4)$ does not belong to* $\mathcal{T}$

The proofs in [12] used some dimension four techniques (Litherland's algorithm [52], Kirby's calculus [47], characteristic classes, old gauge theory).

Remark 1.1. In my Ph. D thesis (see [2]), we also proved that the family $T(p, p+3)$ ($p \geq 5$) does not belong to $\mathcal{T}$.

This let us hit on the following:

CONJECTURE 1.1. *Any non-exceptional torus knot does not belong to* $\mathcal{T}$*.*

Remark 1.2. This conjecture collapsed last Fall (Goda-Hayashi-Song [25]). They proved that $T(p, p+2)$ belongs to $\mathcal{T}$, for any $p \geq 5$ (see [25]), using $(1,1)$-decomposition and Dehn surgery.

Remark 1.3. In [62], K. Miyazaki and K. Motegi showed that if a non-exceptional torus knot $T(p,q)$ $(0 < p < q)$ is obtained from a trivial knot by a single (n,ω)-twisting, then $|n| = 1$. In [12], we actually prove that $n = +1$.

2. Twisting of graph knots [10]

Recall that a knot in S^3 is a graph knot if its exterior is a graph manifold, i.e., there is a family of tori which decompose the exterior $E(k) = S^3 - intN(k)$ into Seifert fiber spaces. Technically, a graph knot is a knot obtained from the unknot by cabling and connected sum operations (e.g. torus knots, iterated torus knots). In [10], we mainly prove the following:

Theorem 2.1. If K_n is a non-exceptional graph knot, then $n = \pm 1$.

By an exceptional graph knot, we mean the special iterated torus knot K_n^m defined as follows:

Definition 2.1 (exceptional pair): Let $K^0 \cup C$ be the Hopf link. Let K^1 be an (ε_1, q_1)-cable of K^0, and K^2 an (ε_2, q_2)-cable of K^1, and similarly K^{i+1} a $(\varepsilon_{i+1}, q_{i+1})$-cable of K^i, where $|\varepsilon_i| = 1$. Then K^m is a trivial knot and K_n^m is an iterated torus knot for any integers m and n; in particular, K_n^1 is a torus knot and if $q_1 = 2$ then $K_{\mp 1}^1$ is a trivial knot. A pair (K, C) is an *exceptional pair* if the link $K \cup C$ is isotopic to a link $K^m \cup C$ for some integer m.

Remark 2.1. Notice that K_n^m is n-twisted for any $n \neq 0$.

We prove Theorem 2.1 by using the following Corrollary:

Corrollary 2.1. Let k be a (non-trivial) prime graph knot in S^3. Every essential planar surface in $E(k)$ whose boundary slope in not $\frac{1}{0}$ is isotopic to a cabling annulus.

Remark 2.2. C. M. Tsau [80] proved the same statement in case k is a non-trivial torus knot.

The proof of Corrollary 2.1 uses Lemma 3.1 in [33], due to C. McA. Gordon and R. A. Litherland ([33]), and JSJ decomposition: Let $\mathcal{T}$ be the set of essential tori in $E(k) = S^3 - \text{int}N(k)$ which gives the torus decomposition of $E(k)$ in the sense of Jaco-Shalen [42] and Johannson [43]. Each component of $E(k) - \mathcal{T}$ is hyperbolic (labeled H) or Seifert fibered; moreover, a Seifert fibred piece is either a torus knot exterior (T), a cable space (Ca), or a composing space (Co) [42]. Figure 4 corresponds respectively to connected sums of two torus knots and two iterated torus knots.

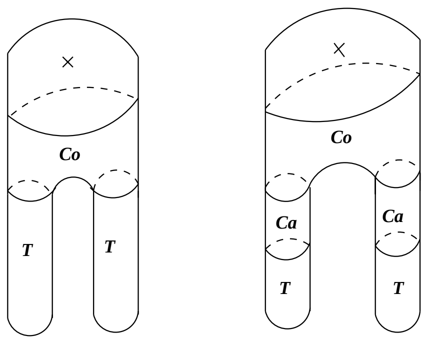

Figure 4:

This problem is included in the general problem of obtaining a Seifert fiber space by Dehn surgery on a knot C in a solid torus V. Lately, Mc. C. Gordon and J. Luecke proved the following ([35]):

If $V(C, \frac{m}{n})$ is toroidal, then $\mid n \mid = 1$ or $V(C, \frac{m}{n})$ is a union of two Seifert spaces. This implies that if a twisted knot K_n is a satellite which is not a cable of a torus knot, then $\mid n \mid = 1$.

3. Twisting of satellite knots and geometric type of twisted knots [13]

Let K_n be a n-twisted knot in S^3, obtained from K along C; and $M = S^3 - intN(K \cup C)$. By W. Thurston's geometrization theorem [79], M is either Seifert fibred, or toroidal, or hyperbolic. The twisting is respectively called Seifert, or toroidal or hyperbolic. We study these cases separately, and the main result of this paper is the following theorem:

Theorem 3.1. If M is hyperbolic and K_n is satellite, then $n = \pm 1$.

Note that $M(-\frac{1}{n}) \cong E(K_n)$ is toroidal and $M(-\frac{1}{0}) \cong S^1 \times D^2$. The proof of Theorem 3.1 is done by studying a pair of graphs of type torus/disk which gives rise to a configuration called *Scharlemann cocycle* which is not well understood. We fully work out this configuration using combinatorial methods such as *white/black Scharlemann cycle*, and dual graphs as well as webs (see [13]).

Some examples of hyperbolic twistings are the following:

Lemma 3.1. If K_n is a non-exceptional torus knot then the twisting is hyperbolic.

Proposition 3.1. Let K_i be simple knots, for $i \in \{1, .., n\}$, then any twisting producing $K_1 \# K_2 ... \# K_n$ is hyperbolic.

Lemma 3.2. If M is Seifert fibred, then K_n is an exceptional torus knot.

Assume M is toroidal and denote by ℓ the core of the separating torus V in S^3. The twisting is said *exotic* if the link $\ell \cup C$ is either one of Mathieu's links with $n = \pm 1$ (Y. Mathieu [56]), or is the Hopf link and n is a positive integer.

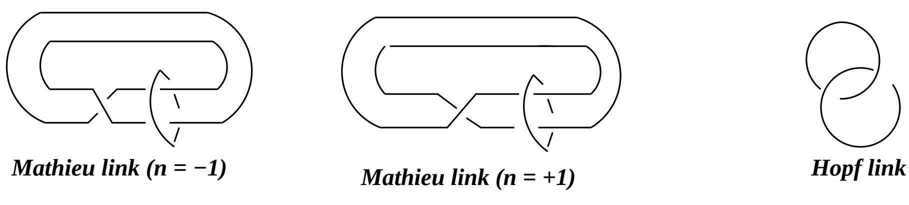

Figure 5:

Corollary 2.1. If M is toroidal and the twising is not exotic then $M(-\frac{1}{n})$ is also toroidal, i.e., K_n is always satellite, for any integer $n \neq 0$.

4. Gromov invariant of twisted knots [11]

Let X be a topological space and $c = \sum_{i=1}^{n} r_i \sigma_i$ be a finite combination of singular k-simplices $\sigma_i : \Delta^l \to X$ with real coefficients r_i. We define the *norm* $\|c\|$ of c by $\sum_{i=1}^{n} |r_i|$. Let M be a compact, orientable, 3-manifold with toral boundary. The *Gromov volume* of M is defined as;

$$\|M\| = \inf \{\|z\|; z \text{ is a singular cycle representing } [M, \partial M]\}$$

Where $[M, \partial M] \in H_3(M, \partial M; \mathbb{R})$ is a fundamental class of $(M, \partial M)$ (see [38]). For a knot K in the 3-sphere S^3, we define the *Gromov volume* of K as that of the exterior $E(K) = S^3 - \text{int}N(K)$ and denote it by $\|K\|$.

Now recall some properties of Gromov volumes [76].

- Let K be a hyperbolic knot, i.e., its complement admits a complete hyperbolic metric. Then

$$\|K\| = \frac{Vol(S^3 - K)}{v_3}$$

  Where $Vol(S^3 - K)$ is the volume of $S^3 - K$ and v_3 is the volume of the regular ideal simplex.

- Let K be a torus knot, i.e., its exterior is a Seifert fiber space, then $\|K\| = 0$.

- Let K be a satellite knot with a family of essential tori $\mathcal{T}$; let P_i $(1 \le i \le n)$ be a component of $E(K) - \mathcal{T}$. Then $\|K\| = \sigma_{i=1}^{n} \|P_i\|$ (T. Soma [76]).

Notice that if (K, C) is an exceptional pair (see Definition 2.1), then $\|K_n\| = 0$ for any integer $n \neq 0$, and that $\|K\|$ is zero if and only if K is a graph knot, i.e., each label appeared at vertices of the satellite diagram is T, Ca or Co (T. Soma [76].

In this paper, we prove the following:

Theorem 4.1. Suppose that K is a trivial knot and (K, C) is not an exceptional pair. Then the Gromov volume of a twisted knot K_n is positive for any integer $|n| > 1$. Moreover, if $\|K_1\| = 0$ (resp. $\|K_{-1}\| = 0$), then $\|K_{-1}\| > 0$ (resp. $\|K_1\| > 0$).

Two major applications which are:

(1) New results regarding the slice genus of (torus) knots in $\mathbb{C}P^2$, and

(2) Counterexamples to two conjectures due to T. Lawson (see [49]).

The genus function G is defined on $H_2(X, \mathbb{Z})$ as follows: For $\alpha \in H_2(X, \mathbb{Z})$, consider

$$G(\alpha) = min\{genus(\Sigma) | \Sigma \subset X \quad represents \quad \alpha, i.e., [\Sigma] = \alpha\}$$

Where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4-manifold X. Note that $G(-\alpha) = G(\alpha)$ and $G(\alpha) \geq 0$ for all $\alpha \in H_2(X, \mathbb{Z})$ (R. E. Gompf and A.I. Stipsicz [29]).

5. The minimal genus problem: New approach [3]

We studied the minimal genus problem of connected sums of 4-manifolds and the minimal slice genus of knots in $\mathbb{C}P^2$. The approach used is twisting operations on knots in S^3. Note that there is no theory for 4-manifolds with even b_2^+ e.g. $\mathbb{C}P^2 \# \mathbb{C}P^2$. Gauge theory is inefficient in this kind of problems because of the vanishing theorem of Seiberg-Witten invariants. To remedy this, there is a connection between knot theory and dimension four which is based on *the gluing of surfaces techniques* as depicted in Figure 6:

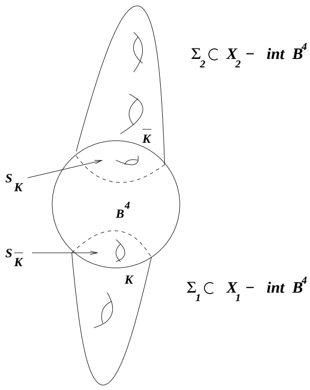

Figure 6:

Let X_1^4 and X_2^4 be two closed 4-manifolds and let $(\Sigma_i, \partial\Sigma_i) \subset (X_i - intB^4, S^3)$ for $i = 1, 2$ be two compact and oriented surfaces such that $\partial\Sigma_1 = K$ and $\partial\Sigma_2 = \overline{K}$, where $\overline{K}$ is the dual knot of K i.e. the inverse of the mirror-image of k. Denote by $\Sigma_1' = \Sigma_1 \bigcup_K S_{\overline{K}}$ and $\Sigma_2' = \Sigma_2 \bigcup_{\overline{K}} S_K$, where S_K (resp. $S_{\overline{K}}$) is the standard Seifert surface for K (resp. $\overline{K}$) in B^4. Gluing Σ_1' and Σ_2' along their boundaries yields a new closed surface $\Sigma_1' \bigcup_K \Sigma_2'$ such that $[\Sigma_1' \bigcup_K \Sigma_2'] = [\Sigma_1] + [\Sigma_2] \in H_2(X_1 \# X_2, \mathbb{Z})$ and

$g(\Sigma_1'\bigcup_K \Sigma_2') = g(\Sigma_1) + g(\Sigma_2)$ (skips the 4-ball genus of K and $\bar{K}$). Let $a = [\Sigma_1'] = [\Sigma_1 \bigcup_K S_{\overline{K}}] \in H_2(X_1,$ and $b = [\Sigma_2'] = [\Sigma_2 \bigcup_{\overline{K}} S_K] \in H_2(X_2, \mathbb{Z})$. Then $a + b = [\Sigma_1 \bigcup_K \Sigma_2] \in H_2(X_1 \# X_2, \mathbb{Z})$.

We give a counterexample to the following conjecture (see Therem 5.3 bellow):

CONJECTURE 5.3 (T. Lawson [49]): *If $X = X_1 \# X_2$ is the connected sum of two symplectic 4-manifolds with $b_2^+ \geq 3$, and if $(a, b) \in H_2(X) = H_2(X_1) \oplus H_2(X_2)$ satisfies $a.a \geq 0$, $b.b \geq 0$, then the minimal genus for this class is the sum of the minimal genus for the class a and the minimal genus for the class b.*

Let $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ be the 4-manifold equipped with an elliptic fibration, and denote $E(2) = E(1)\#_f E(1)$ be the fiber sum (see R. Gompf and A. Stipsicz [31], pp.67 − 76 for more details on elliptic fibrations). We can check that $E(2)$ is a $K3$ surface and then $b_2^+ = 3$ and $b_2^- = 19$.

Theorem 5.3. *There exist $(a, b) \in H_2(E(2)\#E(2)) = H_2(E(2)) \oplus H_2(E(2))$ such that $a.a \geq 0, b.b \geq 0$, and the genus of a (resp. b) is minimal in $H_2(E(2))$ (resp. $H_2(E(2))$, but the genus of $a + b$ is less and not equal to the sum of the genus of a and the genus of b.*

Sketch of a proof: We find a properly embedded surface in $E(2) - intB^4$ with boudary the figure-eight knot 4_1 and such that its closure is genus-minimizing in $E(2)$. If we glue two copies of this surfaces, we skip the 4-ball genus of 4_1, and then $G(a + b) = G(a) + G(b) - 2$, which contradicts the symplectic Lawson's conjecture (see [3] for more details).

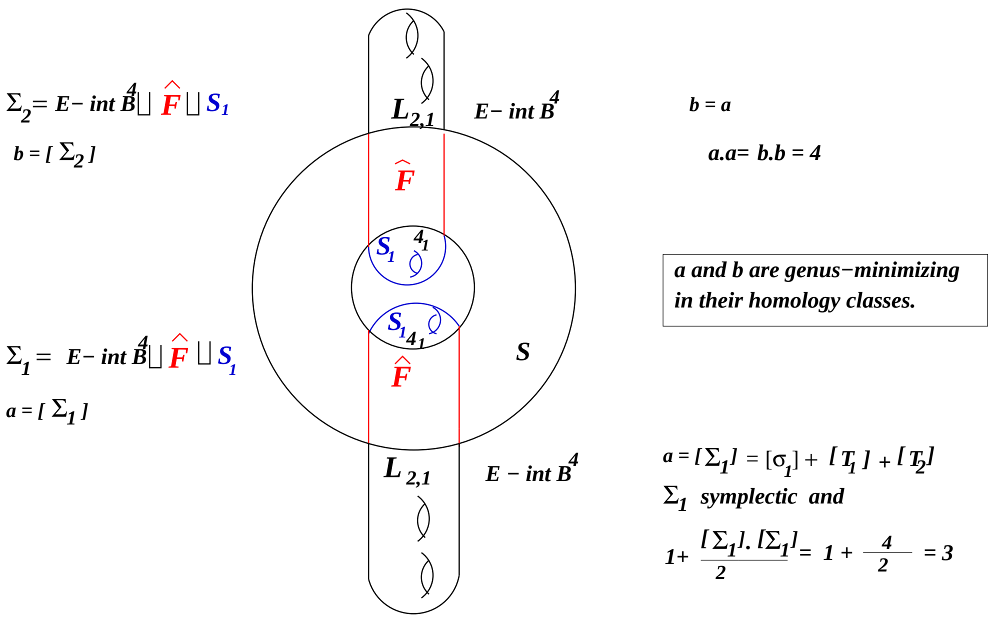

Figure 7:

6. Genus and degrees of torus knots in $\mathbb{C}P^2$ [3] and [5]

We study surfaces in $\mathbb{C}P^2 - intB^4$ bounding knots in $\partial(\mathbb{C}P^2 - intB^4) \cong S^3$ in general (Fig. 8).

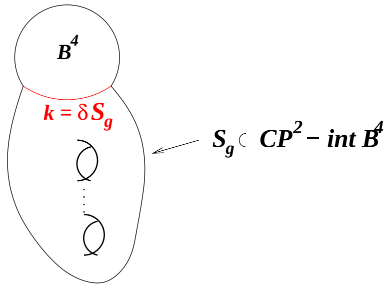

Figure 8:

We call an orientable and compact surface $(S, \partial S) \subset (\mathbb{C}P^2 - intB^4, S^3 \cong \partial(\mathbb{C}P^2 - intB^4))$ such that $\partial S = k$ a *Seifert surface* for k. Let $g_s(K)$ be the minimal genus over all isotopy classes of Seifert surfaces of k. There exist $d \in \mathbb{Z}$ such that $[S] = d\gamma$ where γ is the standard generator of $H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z})$. For simplicity, we call d (resp. $g_s(K)$) the degree (resp. the genus) of k in $\mathbb{C}P^2$.

We hit the following question:

Question 6.1. *What is the smooth slice genus of a $(\pm p, q)$-torus knot in $\mathbb{C}P^2 - intB^4$?*

Note that $T(p, q)$ is obtained from $T(2, 3)$ by adding $(p-1)(q-1)-2$ half-twisted bands. This implies that there is a genus $\dfrac{(p-1)(q-1)-2}{2}$ concordance between $T(2, 3)$ and $T(p, q)$. We claim that the smooth slice genus in $\mathbb{C}P^2$ and the concordance genus are the same for any (p, q)-torus knot ($0 < p < q$ and p and q are coprime). This let us hit to the following conjecture:

Conjecture 6.1 $g_s(T(p, q)) = \dfrac{(p-1)(q-1)}{2} - 1.$

In [3], we proved the following:

Theorem 6.1. $g_s(T(-p, q)) \leq \dfrac{(q-1)(q-p-1)}{2}$.

Sketch of the proof: To prove Theorem 6.1, we gave an explicite Seifert surface for $T(-p, q)$ in $\mathbb{C}P^2 - intB^4$, using Milnor fiber [60] and the positive answer to Thom's conjecture [48] (see Claim 2.1 in [3] for a proof). We can easily conclude from Theorem 6.1 that $T(-p, p+1)$ is slice in $\mathbb{C}P^2$.

The smooth slice genus of non-slice torus knots in $\mathbb{C}P^2$ is still unknown. To solve this problem, we give the smooth slice genus and the possible degrees in $\mathbb{C}P^2$ for all $(\pm 2, q)$-torus knots ($3 \leq q \leq 11$). For this, we proved the following:

Lemma 6.1. Let d be an odd prime number. Then the d-signature of $(2, q)$-torus knots ($q \geq 3$) is given by the formula:

$$\sigma_d((T(2,q)) = -(q-1) + 2[\frac{q}{2d}]$$

Where $[x]$ denotes the greatest integer less or equal to x.

In Theorem 6.2 and Theorem 6.3 stated bellow and proved in [5], we say that the degree d of a knot k is *realizable* if a minimal genus Seifert surface $(S, \partial S) \subset (\mathbb{C}P^2 - intB^4, S^3)$ for the knot $k \subset \partial(\mathbb{C}P^2 - intB^4) \cong S^3$ can be given explicitely e.g. $(-1, \omega)$-twisted knots have their degrees $d = \omega$ realizable by the twisting disks.

Theorem 6.2.

(1) $g_s(T(-2,3)) = 0$ *with realizable degree* $d \in \{\pm 2, \pm 3\}$.

(2) $g_s(T(-2,q)) = 0$ *for* $q = 5, 7$ *and* 9 *with respective realizable degrees* $\pm 3, \pm 4$ *and* ± 4.

(3) $g_s(T(-2,11)) = 1$ *with degree* $d \in \{\pm 4, \pm 5\}$.

It would be interesting to find the exact degree (may be all the degrees are possible) of $T(-2, 11)$ and show that it is (or they are) realizable with a torus.

A plausible conjecture is the following:

Conjecture 6.2. $g_s(T(-2,q)) = \dfrac{q-9}{2}$ for any $q \geq 11$.

Theorem 6.3.

(1) $g_s(T(2,3)) = 0$ *with realizable degree* $d = 0$.

(2) $g_s(T(2,q)) = \dfrac{q-3}{2}$ *for* $5 \leq q \leq 11$ *with* $d \in \{0, \pm 1\}$ *if* $q \in \{5, 7, 9\}$ *and* $d \in \{0, \pm 5\}$ *if* $q = 11$.

A plausible conjecture is the following:

Conjecture 6.3. $g_s(T(2,q)) = \dfrac{q-3}{2}$ for any $q \geq 3$.

The proofs use gauge theory and twisting operations on knots. By using various techniques derived from gauge theory e.g. Ozsvath-Szabo knot invariant [68], Gilmer-Viro inequality (P. Gilmer [24], O. Ya. Viro [87]), Acosta-Fintushel-Yasuhara inequality (D. Acosta [1], R. Fintushel [19], A. Yasuhara [86]), T. Kawamura's work (see [44] and [45]).

7. The minimal genus problem in $\mathbb{C}P^2\#\mathbb{C}P^2$ [4] and [3]

By the positive answer to Thom conjecture due to Kronheimer and Mrowka [48], it is known that an algebraic curve of degree d in $\mathbb{C}P^2$ is genus-minimizing and its genus is $\binom{|d|-1}{2} = \dfrac{(\mid d \mid -1)(\mid d \mid -2)}{2}$.

T. Lawson tried to generalize this conjecture to $\mathbb{C}P^2\#\mathbb{C}P^2$.

Conjecture 7.1 (T. Lawson [49]): the minimal genus of $(m,n) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$ is given by $\binom{m-1}{2} + \binom{n-1}{2}$ -this is the genus realized by the connected sum of algebraic curves in each factor.

We give a conterexample to this conjecture by showing the following [3]:

Proposition 7.1. There exist $d \in \{0, \pm 1\}$ such that Lawson's conjecture fails for the pair $(4,d) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$.

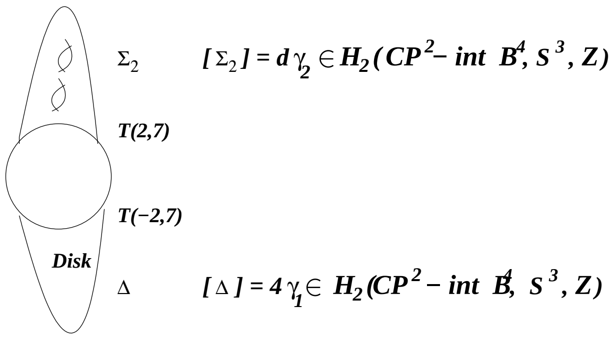

Figure 9:

Sketch of a proof: By gluing two surfaces bounding respectively $T(-2,7)$ and $T(2,7)$ in two copies of $\mathbb{C}P^2 - intB^4$, we get a closed surface in $\mathbb{C}P^2\#\mathbb{C}P^2$ that gives a couterexample to Lawson's conjecture. Indeed, we showed in [3] that $g_s(T(2,7) = 2$ and $d \in \{0, \pm 1\}$, and $T(-2,7)$ is $(-1,4)$-twisted which implies that $T(-2,7)$ is smoothly slice in $\mathbb{C}P^2$ with degree 4. Therefore, the genus-two smooth surface $\Sigma = \Delta \cup \Sigma_2$ in $\mathbb{C}P^2\#\mathbb{C}P^2$ satisfies $[\Sigma] = 4\gamma_1 + d\gamma_2 \subset H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ with $d \in \{0, \pm 1\}$ (see Figure 9). If Lawson's conjecture were true, then the genus of Σ (which is two) should be greater or equal to the proposed Lawson's minimal genus value for the pair $(4,d) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ which is $3 + \frac{(|d|-1)(|d|-2)}{2}$ where $d \in \{0, \pm 1\}$, a contradiction.

In [4], we answer this conjecture by the positive for the small pairs $(3,3)$ and $(6,6)$. The proofs use twisting of knots in S^3 and gauge theory. We also give an explicite representative for $(2,2n) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$ for any $n \geq 1$ whose genus is the proposed Lawson's minimal genus value.

Theorem 7.1. *The minimal genus of the pairs* $(3,3)$ *and* $(6,6) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2)$ *are respectively* 2 *and* 20.

To prove these theorems, part of the proofs was inspired from gauge theory such as in Gilmer-Viro ([24],[87]), Kronheimer-Mrowka [48], Morgan-Szabo-Taubes [58], D. Auckly [16] and T. Kawamura ([44], [45]).

8. *Characteristic twisting invariant of knots in* S^3 *[7]*

Characteristic classes (see Milnor-Stasheff book [61]) play an important role in dimension four topology and geometry. A class $\xi \in H_2(X, \mathbb{Z})$ is characteristic class if ξ is dual to the second Stiefel-Whitney class $w_2(X^4) \in H_2(X, \mathbb{Z})$. An equivalent algebraic definition is the following:

Definition 8.1. $\xi = \sum_{i=1}^{i=n} a_i\gamma_i \in H_2(X, \mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X, \mathbb{Z})$, where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index.

We introduced the following definition in [7]:

Definition 8.2. *A twisting is called characteristic if the twist disk is characteristic*

We introduced a new invariant of knots, derived from twisting, that I called *characteristic twisting invariant*, motivated by characteristic classes. The main idea of this invariant is to look to the homology of the characteristic twist disk bounded by a knot, obtained by a series of characteristic twistings, in a punctured standard 4-manifold i.e. of the form $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}\#S^2 \times S^2 - intB^4$, and derive an invariant from this disk (see 8). It is defined for any integer $n \geq 0$ as follows:

Definition 1.1. *Let K be a smooth slice knot in B^4, and k a knot in S^3 such that there exist a positive integer m and a family of Dehn disks $D_1, D_2, ..., D_m$, along which, we perform successively a series of (n_i, ω_i)-twisting:*

$$K \overset{(n_1,\omega_1)}{\longrightarrow} K(n_1, \omega_1)... \overset{(n_m,\omega_m)}{\longrightarrow} K((n_1, \omega_1), ..., (n_m, \omega_m)) = k.$$

The characteristic invariant of the isotopy class of k, denoted by $ch_n(k)$, is the minimum over all such integers m among all diagrams of k with the following conditions:

(1) *For any $i \in \{1, 2, ...m\}$ we have $| \omega_i | \leq n$.*

(2) *There exist at least $i \in \{1, 2, ...m\}$ such that $| \omega_i |= n$.*

(3) $n_i \neq 0$ *for any* $i \in \{1, 2, \cdots, m\}$.

(4) *If ω_i is even for some $i \in \{1, ..., m\}$, then n_i is even.*

$ch_n(k) = 0$ if either the four conditions above are not simultaneousely met; for any $\{(n_i, \omega_i)\}_{i=1}^{i=n}$, or k is a smoothly slice knot.

Definition 2. *The characteristic twisting polynomial of degree $N \geq 0$ of a knot k in S^3 is a Laurent polynomial in a variable X:*

$$Ch_N(k) = \sum_{n=0}^{n=N} ch_n(k)X^n$$

DEFINITION 3. *The characteristic twisting power series expansion of a knot k in S^3 is the Laurent series:*

$$Ch(k) = \sum_{n=0}^{\infty} ch_n(k) X^n$$

The second homology of the disk bounded by such a knot k, obtained by a series of twistings, is a characteristic class (assured by condition (1)) in a standard 4-manifold which derive an invariant ch_n of the knot k (see [7]), for any positive integer n. This invariant is a concordance invariant and is sometimes finer than the Alexander and the HOMFLY polynomials e.g. the invariant ch_3 is able to distinguish between $3_1\#3_1$ and $3_1\#\bar{3}_1$ unlike the Alexander polynomial since $ch_3(3_1\#\bar{3}_1) = 0$ and $ch_3(3_1\#3_1) = 2$, and also to distinguish between 5_1 and 10_{132} unlike the HOMFLY since $ch_3(5_1) = 1$ and $ch_3(10_{132}) = 2$. This invariant can also tells knots apart e.g. ch_3 distinguishes between the trefoil and the figure-eight knot since $ch_3(3_1) = 1$ and $ch_3(4_1) = 2$. The following questions are still open:

- *Can Ch distinguish between mutants ?*
- *Is there a variation of Ch that can detect amphicheirality ?*
- *Is Ch a* 1-1 *invariant modulo concordance ?*

CURRENT RESEARCH

1. Slice genus and degree of torus knots in $\mathbb{C}P^2$

An interesting question is to find the smooth slice genus and the degree of torus knots in $\mathbb{C}P^2$ in general. Note that $T(p,q)$ is obtained from $T(2,3)$ by adding $(p-1)(q-1)-2$ half-twisted bands. This let us hit to the following conjecture:

CONJECTURE 1.1 $g_s(T(p,q)) = \dfrac{(p-1)(q-1)}{2} - 1.$

2. Torus knots and twisting

I believe that the following conjectures are true:

CONJECTURE 2.1. *A (p,q)-torus knot is twisted iff $q = kp \pm 1$ or $q = p+2$ ($k > 0$ and $p \geq 7$).*

CONJECTURE 2.2. *A torus knot is a slice torus knot in $\mathbb{C}P^2$ iff it is (-1)-twisted.*
This conjecture 2.2 is true for slice non-exceptional torus knot and $T(-p, p+2)$ $(p \geq 2)$ (see [25]).

3. The number of triple points of 2-knots

A surface-link is a 2-dimensional manifold embedded locally flatly in $\mathbb{R}^4$. It is called a surface-knot when F is connected and a 2-knot when F is homeomorphic to a 2-sphere. To describe a surface-knot we use its projection $\pi : \mathbb{R}^4 \mapsto \mathbb{R}^3$. The set of singular points in the generic projection image $\pi(F)$ is consisted of isolated branch points, double point curves, and isolated triple points. If there is no branch point, then it is an immersion. The minimal triple point number $t(F)$ is the smallest number of triple points among all isotopy classes of the diagrams of the surface-knots with the same type as F (see A. Satoh [70] for more details). This definition is an analogy of the "minimal crossing number" in classical knot theory. An interesting class of 2-knots are the m-twist-spun knots introduced by Zeeman. S. Shin and A. Satoh proved in [70], that $t(\tau^2 3_1) = 4$, where $\tau^2 3_1$ denotes the 2-twist-spun of the trefoil knot. E. Hatakenaka proved in [39] that $6 \leq \tau^2 4_1 \leq 8$, where 4_1 is the figure-eight knot. In a joint work with M. Elhamdadi (see [8]), we are trying to classify all surface-knots with minimal triple point number 4 (we conjecture that there is only one such surface which is ambiant isotopic to $\tau^2 3_1$). We hit with the following conjectures:

Conjecture 3.1. $t(\tau^2 k)$ is always a multiple of 4, for any knot k.

Conjecture 3.2. $t(\tau^2 4_1) = 8$.

4. Lawson's conjecture on the minimal genus problem

T. Lawson asked the following question, that I am hoping to give an answer:

Question 4.1. *Can the homology* $(3, 2) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ *be represented by a sphere ?*

The general Lawson's conjecture for any pair $(m, n) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2)$ is also under investigation.

FURTHER RESEARCH

1. Composite Knots with more than two factors

C. Hayashi and K. Motegi [64], Goodman-Strauss [31], and M. Teragaito [77] gave examples of composite knots with two prime factors that are twisted, or equivalentely can be untied by one twisting. However, the number of prime factors of such knots does not exceed two. They suggest the following

Conjecture 1.1: composite knots with more than two prime factors are non-twisted.

Combinatorial methods as in CGLS [17] and Floar homology are possible tools.

2. Topological but not smoothly slice knots in $\mathbb{C}P^2$:

It is known via gauge theory and Freedman's work that many topologically slice knots are not smoothly slice in the 4-ball. For example, any knot with trivial Alexander polynomial e.g. the untwisted double of any knot [20]. In particular, R. Gompf showed in his thesis that the untwisted double of the right-handed trefoil is not smoothly slice (see [28]). I am working on the problem of finding examples of knots which are topologically but not smoothly slice in $\mathbb{C}P^2$.

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