Math 240
Linear Algebra
Homework 2

1. Find the inverse of each matrix.

(**a**) $A = \begin{bmatrix} 5 & 8 \\ -8 & 7 \end{bmatrix}$ *Solution.* $\frac{1}{99}\begin{bmatrix} 7 & -8 \\ 8 & 5 \end{bmatrix}$

(**b**) $B = \begin{bmatrix} 2-2\mathrm{i} & -5-2\mathrm{i} \\ 1 & -1-2\mathrm{i} \end{bmatrix}$ *Solution.* $\begin{bmatrix} 1+2\mathrm{i} & -5-2\mathrm{i} \\ 1 & -2+2\mathrm{i} \end{bmatrix}$

(**c**) $C = \begin{bmatrix} -4 & 12 & 35 \\ 1 & -4 & -12 \\ -1 & 3 & 9 \end{bmatrix}$ *Solution.* $\begin{bmatrix} 0 & -3 & -4 \\ 3 & -1 & -13 \\ -1 & 0 & 4 \end{bmatrix}$

(**d**) $D = \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ *Solution.* $\begin{bmatrix} 1 & 4 & -9 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find all values of k for which the matrix

$$A = \begin{bmatrix} 2 & -7 \\ 5 & k \end{bmatrix}$$

is invertible.

Solution. By Theorem 2.3.6, $\det A \neq 0$ if and only if A is invertible. $\det A = 2k+35$. If $\det A = 0$, then $0 = 2k + 35$. This implies that A is not invertible for $k = \frac{-35}{2}$. And so, A is invertible for $k \neq \frac{-35}{2}$.

3. Find the determinant for each matrix.

(**a**) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Solution. Note that A is just the identity matrix with two exchanged rows. Because the identity matrix is a triangular matrix, we know by theorem 2.1.3(b) that $\det I = 1$. Hence, by theorem 2.2.4(b), $\det A = -(-\det I) = 1$.

(b) $B = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Solution. $a^2(c-b) + b^2(a-c) + c^2(b-a)$. Note that with a little more effort one can write the solution as $(b-a)(c-a)(c-b)$.

(c) $C = \begin{bmatrix} 2 & 2 & -9 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$

Solution. Notice C is a triangular matrix. By theorem 2.1.3(b), $\det C = 2 \cdot 1 \cdot 5 = 10$.

(d) $D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 8 & 9 & 1 & -3 \end{bmatrix}$

Solution. Notice D is a triangular matrix. By theorem 2.1.3(b), $\det C = -1 \cdot 2 \cdot -1 \cdot -3 = -6$.

(e) $E = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & -3 & 0 & 0 \end{bmatrix}$

Solution. $\det E = 30$

(f) $F = \begin{bmatrix} 2 & -5 \\ -11 & -8 \end{bmatrix}$

Solution. $\det F = -16 - 55 = -71$

4. Suppose that A and B are 4×4 matrices, where $\det(A) = 1$ and $\det(B) = 8$. Find

(a) $\det(AB)$ *Solution.* $\det(AB) = \det A \det B = 1 \cdot 8 = 8$

(b) $\det(2A)$ *Solution.* $\det(2A) = 2^4 \det A = 16 \cdot 1 = 16$

(c) $\det\left(A^\top\right)$ *Solution.* $\det\left(A^\top\right) = \det A = 1$

(d) $\det\left(B^{-1}\right)$ *Solution.* $\det\left(B^{-1}\right) = \left(\dfrac{1}{\det B}\right) = \dfrac{1}{8}$

(e) $\det\left(B^5\right)$ *Solution.* $\det\left(B^5\right) = (\det B)^5 = 8^5 = 32,768$

5. If

$$A = \begin{bmatrix} -3e^{3t} & 3e^{4t} \\ 2e^{3t} & 3e^{4t} \end{bmatrix}$$

then find

(**a**) $\det(A)$ *Solution.* $\det(A) = -15e^{7t}$

(**b**) $\operatorname{adj}(A)$ *Solution.* $C^T = \begin{bmatrix} 3e^{4t} & -3e^{4t} \\ -2e^{3t} & -3e^{3t} \end{bmatrix}$

(**c**) A^{-1} *Solution.* $A^{-1} = \dfrac{1}{\det A}\operatorname{adj}(A) = -\dfrac{1}{15e^{7t}} \begin{bmatrix} 3e^{4t} & -3e^{4t} \\ -2e^{3t} & -3e^{3t} \end{bmatrix} = \dfrac{1}{15} \begin{bmatrix} -3e^{-3t} & 3e^{-3t} \\ 2e^{-4t} & 3e^{-4t} \end{bmatrix}$

(**d**) C: the matrix of cofactors. *Solution.* $C = \begin{bmatrix} 3e^{4t} & -2e^{3t} \\ -3e^{4t} & -3e^{3t} \end{bmatrix}$

6. Use the adjoint of the matrix to find the inverse for each matrix.

(**a**) $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix}$

Solution. $\operatorname{adj}(A) = \begin{bmatrix} -3 & -1 & 9 \\ 6 & -10 & 18 \\ 6 & 14 & -18 \end{bmatrix}$. So $A^{-1} = \dfrac{1}{36} \begin{bmatrix} -3 & -1 & 9 \\ 6 & -10 & 18 \\ 6 & 14 & -18 \end{bmatrix}$

(**b**) $B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Solution. $\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. So $A^{-1} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

7. Solve using Cramer's Rule.

(**a**) $\begin{cases} 6x_1 - 3x_2 = 39 \\ 5x_1 + 9x_2 = -25 \end{cases}$

Solution. $x = \dfrac{|D_x|}{|D|} = \dfrac{\begin{vmatrix} 39 & -3 \\ -25 & 9 \end{vmatrix}}{\begin{vmatrix} 6 & -3 \\ 5 & 9 \end{vmatrix}} = \dfrac{276}{69} = 4$

$$y = \frac{|D_y|}{|D|} = \frac{\begin{vmatrix} 6 & 39 \\ 5 & -25 \end{vmatrix}}{\begin{vmatrix} 6 & -3 \\ 5 & 9 \end{vmatrix}} = \frac{-345}{69} = -5$$

(**b**) $\begin{cases} -x_1 + 2x_2 + 7x_3 = 13 \\ 2x_1 - x_2 - 2x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = -14 \end{cases}$

Solution. $x_1 = \dfrac{|D_{x_1}|}{|D|} = \dfrac{0}{63} = 0$, $x_2 = \dfrac{|D_{x_2}|}{|D|} = \dfrac{-252}{63} = -4$, and $x_3 = \dfrac{|D_{x_3}|}{|D|} = \dfrac{189}{27} = 3$

8. Find all values of λ for which $\det(A) = 0$.

(**a**) $A = \begin{bmatrix} \lambda - 4 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

Solution. $\det A = (\lambda - 4)(\lambda + 4) + 5 = \lambda^2 - 16 + 5 = \lambda^2 - 11 = 0$. This implies $\lambda^2 = 11$. Thus, $\lambda = \pm\sqrt{11}$.

(**b**) $A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 3 \end{bmatrix}$

Solution. $\det A = (\lambda - 4)[\lambda(\lambda - 3) - 6] = 0$. This implies $(\lambda - 4) = 0$ and $\lambda^2 - 3\lambda - 6 = 0$. Hence, $\lambda = 4$ and using the quadratic formula we get $\lambda = \dfrac{3 \pm \sqrt{33}}{2}$. So the values for λ for which $\det(A) = 0$ are $\lambda = 4, \dfrac{3 \pm \sqrt{33}}{2}$.