Math 240
Linear Algebra
Homework 3

1. True or false? If you answer false, provide a counter example or simply explain your answer.

(**a**) Every vector space contains a zero vector.

(**b**) A vector space may have more than one vector.

(**c**) If f and g are polynomials of degree n, then $f+g$ are of degree n.

(**d**) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(**e**) The empty set is a subspace of every vector space.

(**f**) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(**g**) The zero vector is a linear combination of any nonempty set of vectors.

(**h**) The span of $\emptyset$ is $\emptyset$.

(**i**) Any set containing the zero vector is linearly dependent.

(**j**) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \cdots, x_n$ are linearly independent, then all scalars a_i are zero.

2. Let $V = \big\{(a_1, a_2) \colon a_1, a_2 \in \mathbb{R}\big\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space? Why or why not?

3. Let V and W be vector spaces over an field F. Suppose that

$$Z = \big\{(v, w) \colon v \in V \text{ and } w \in W\big\}.$$

Prove that Z is a vector space over F under the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + w_1, v_2 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

4. Determine of the following span $\mathbb{R}^3$

(**a**) $(2, 2, 2), (0, 0, 3), (0, 1, 1)$

(**b**) $(2, -1, 3), (4, 1, 2), (8, -1, 8)$

5. Which of the following sets are linearly dependent.

(**a**) $(4, -1, 2), (-4, 10, 2)$

(**b**) $(-3, 0, -4), (5, -1, 2), (1, 1, 3)$

(**c**) $2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2$

(**d**) $3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2$

6. Determine whether or not the first vector (polynomial) can be expressed as a linear combination of the other two. If it can, write the linear combination.

(**a**) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$

(**b**) $(1, 2, -3), (-3, 2, 1), (2, -1, -1)$

(**c**) $x^3 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1$

(**d**) $4x^3 + 2x^2 - 6, x^3 - 2x^2 + 4x + 1, 3x^3 - 6x^2 + x + 4$

7. Prove that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are linearly independent in $M_{2\times 2}(F)$.