Quiz 1. Show that the sequence

\[
\begin{array}{cccccc}
1 & 3 & 7 & 15 & 31 \\
2 & 4 & 8 & 16 & 32 & \ldots
\end{array}
\]

converges to 1.
Solution. Let $\varepsilon > 0$. Note that in this sequence, $a_n = 1 - \frac{1}{2n}$. Pick $N \in \mathbb{N}$ so that $N > \log_2(\frac{1}{\varepsilon})$. Now let $n \geq N$. Then

$$|1 - a_n| = \frac{1}{2n} \leq \frac{1}{2N} < \frac{1}{2\log_2(\frac{1}{\varepsilon})} = \frac{1}{\frac{\varepsilon}{\varepsilon}} = \varepsilon.$$ 

Thus $a_n \to 1$.

Key items to look for.

- Letting $\varepsilon > 0$ be a generic value larger than 0. By proving it for a generic value, then the result holds for all values, and this satisfies the “for every” part of the definition.

- Picking some value of $N \in \mathbb{N}$ for which we will show the conclusion. Note that in the above proof the value of $N$ is optimal, but all you need is to find a value of $N$ that works; it doesn’t have to be optimal.

- Showing that if you let $n \geq N$ then $d(a_n, 1) < \varepsilon$. Sometimes this is a straightforward calculation; sometimes it isn’t.

Often, to find a value of $N$, it is helpful to work backward from what we need $N$ to satisfy, i.e. we want $|a_n - 1| < \varepsilon$, so

$$|a_n - 1| < \varepsilon$$

$$\Leftrightarrow 1 - 2n < \varepsilon$$

$$\Leftrightarrow -n \log_2 2 < \log_2 \varepsilon$$

$$\Leftrightarrow n > -\log_2 \varepsilon = \log_2 \frac{1}{\varepsilon}.$$ 

Therefore, if we let $n > \log_2 \frac{1}{\varepsilon}$, then $a_n$ is within $\varepsilon$ of 1. In particular, if we let $N > \log_2 \frac{1}{\varepsilon}$ and let $n \geq N$, then this satisfies what we want.