Math 140 Final Review
Monday May 12 1:00-3:00 for 12:00 class
Friday May 16 4:00-6:00 for 3:00 class

The final will be comprehensive over the material in class.
This review is designed to be an aid in study for the final. It is not designed to mimic exactly what will be on the exam. The problems on the exam may be different than those in this document. The ideas, however, which are used here will be of great use.

Here are some useful general things to know:

• Algebra, functions, and graphs
  – Definition of function, domain, range
  – Graphs of functions, vertical line test, shifting and reflecting graphs, piecewise-defined functions
  – Linear functions, slope, intercepts, finding the equation of a line (point-slope formula, etc.)
  – Quadratic functions, intercepts, location of vertex, graphing a quadratic function
  – Polynomials, degree, roots, turning points
  – Rational functions, intercepts, asymptotes, sketching a graph
  – Exponentials, properties of exponentials (p.98-99 and others we discussed in class), exponential equations, graphs of exponential functions, 1-1 functions (and horizontal line test), $e^x$, compound interest and population growth
  – Logarithms, rewriting exponentials, properties of logarithms (p.114-115 and others we discussed in class), logarithmic expressions, solving exponential and logarithmic equations
  – Limits, computing limits, limit properties (p.140), estimating limits from graphs (such as exercises 3.1.25-28), limit of a difference quotient
  – Continuous functions, definition of continuity, identifying continuous functions and discontinuities from graphs, continuity properties (p.153), asymptotes and continuity, drawing a graph with certain limit and continuity properties

• Derivatives, tangent lines, and rates of change
  – Limit properties of functions
  – Continuity
  – Average rate of change
- Definition of derivative and finding the derivative of a function from the definition

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

- Instantaneous rate of change
- Slope of the tangent line
- Derivatives of various forms of functions:
  * sums and differences
    \[ (f + g)' = f' + g' \]
    \[ (f - g)' = f' - g' \]
  * constants
    \[ \frac{d}{dx}(c) = 0 \]
  * The power rule
    \[ \frac{d}{dx}(x^n) = nx^{n-1} \]
  * a constant times a function
    \[ (c \cdot f)' = c \cdot f' \]
  * the product rule
    \[ (fg)' = f'g + fg' \]
  * the quotient rule
    \[ \left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2} \]
  * chain rule: power form
    \[ (f^n)' = nf^{n-1} \cdot f' \]

- Finding the tangent line to a function at a point.
- Estimating function values from the tangent line (compare with marginal cost)
- Graphs and derivatives
- Sign of the \( f'(x) \leftrightarrow f(x) \) increasing/decreasing
- Maxima/minima/saddle points
- Critical points and optimization
- Second derivatives and graphs
- Curvature and concavity
- Second derivative test for extrema
- Derivatives of \( \ln x \) and \( e^x \)
• Antiderivatives and integrals
  
  – Definition of antiderivative: If \( f(x) \) is a function, an antiderivative \( F(x) \) is a function such that
    \[
    F'(x) = f(x)
    \]
  
  – If \( F(x) \) is an antiderivative for \( f(x) \), then so is \( F(x) + C \) for any constant \( C \).
  
  – Indefinite integrals:
    \[
    \int f(x) \, dx = F(x) + C,
    \]
    where \( F(x) \) is an antiderivative of \( f(x) \)
  
  – Reconstructing a function from its derivative
  
  – Antiderivatives of \( \frac{1}{2} x \) and \( e^x \)
  
  – The view that we’re adding up things at the rate of the height of the integrand (recall the area model of integration in class)
  
  – Definite integrals
  
  – The Fundamental Theorem of Calculus:
    \[
    \int_a^b f(x) \, dx = F(b) - F(a),
    \]
    where \( F(x) \) is an antiderivative of \( f(x) \).
    In other words,
    \[
    \int_a^b s'(x) \, dx = \text{total change in } s(x) \text{ between } x = a \text{ and } x = b
    \]

Here are some exercises to help in your studies.

1. Given the graph of the function \( f(x) = x^2 \), sketch the graph of \( y = -f(x + 1) - 2 \).

2. Sketch the graph of

   \[
   g(x) = \begin{cases} 
   0, & x \leq 0 \\
   x^2, & 0 < x \leq 1 \\
   2x - 1, & x > 1 
   \end{cases}
   \]

3. In problems 1.1.7-12 in the textbook, which ones are graphs of functions? Explain your reasoning.

4. Find the equation of the line through the points \((-3, 5)\) and \((8, -2)\).
5. Given the quadratic equation

\[ h(x) = 2x^2 + 8x + 6, \]

(a) find the \( x \)-intercepts (if they exist),
(b) the \( y \)-intercept,
(c) the coordinates of the vertex,
(d) and sketch the graph of this function.
(e) Without actually finding the \( x \)-intercepts, how can you find out if there are 0, 1, or 2 intercepts?

6. Answer the questions for problems 2.1.7-14

7. What is the most number of roots that this polynomial can have? Why?

\[ p(x) = 3x^7 + \pi x^{32} - x^2 + 17022x^{1001} \]

8. Consider the function

\[ r(x) = \frac{x - 1}{2x + 1}. \]

(a) What are the \( x \)-intercept(s)?
(b) What are the \( y \)-intercept(s)?
(c) Find the horizontal and vertical asymptotes.
(d) Using your information, sketch a graph of this function.

9. Solve the equation

\[ 3^{5x+5} = 9^{3x+1}. \]

10. You invest $2000 in an investment which pays 24% interest compounded monthly. How much will you have after 8 years?

11. Combine so that you express this as a single logarithm:

\[ 2 \log_5 x + \log_5 7 - 3 \log_5 y + 1 \]

12. Evaluate the following expressions

(a) \( \log_3 9 \)
(b) \( \log_4 32 \)
(c) \( \log_5 5^2 9 \)
13. Solve the equations
   (a) \[4^{3x-1} = 50\]
   (b) \[3^{x+1} = 5^x\]
   (c) \[\log_5(x - 1) + \log_5(x + 3) = 1\]

14. Suppose you invest $5000 in an account paying 12% interest compounded quarterly, how long will it take for you to accumulate $30000?

15. If you invest $1500 in an account with continuously compounding interest, and it takes 20 years to get a return $15,000, what is the rate of interest you are getting on the account? (Note this problem is like the problems economists solve when they talk about the average rate of growth of aspects of the economy such as rate of growth of the GDP over the past, say, 20 years.)

16. Suppose you invest $3000 in an account paying 10% interest compounded continuously. How long will it take to have a balance of $10000?

17. Using the properties of limits, find \[\lim_{x \to 3} \sqrt{4x + 4}.\]

18. Consider the function \[f(x) = \frac{x^2 - 9}{x - 3}.
   (a) Using the properties of limits, find \(\lim_{x \to 3} f(x)\), if it exists.
   (b) Find \(f(3)\), if it exists.
   (c) Does \(\lim_{x \to 3} f(x) = f(3)\)? If not, can you redefine \(f(3)\) so that it does?

   (a) For which \(x\) values in the domain is this function discontinuous?
   (b) For these \(x\) values you found, for which ones can you define or redefine \(f(x)\) at that point so that the function is continuous.
   (c) For which points \(x = c\) will the function always be discontinuous, regardless of the value of \(f(c)\)?

20. Sketch a graph of a function \(y = g(x)\) which has these properties:
(a) \( \lim_{x \to 2^-} g(x) = 1 \)
(b) \( \lim_{x \to 2^+} g(x) = 3 \).
(c) \( g(2) = -1 \).
(d) \( g(x) \) is continuous for all \( x \neq 2 \).
(e) \( g(-2) = 4 \).

(Note: There are a lot of possible solutions to this question.)

21. Let \( g(x) = x^2 - 3x + \sqrt{x} \). What is the average rate of change of \( g(x) \) between \( x = 1 \) and \( x = 9 \)?

22. Let \( P(t) = t^2 - 12t - 20 \) be the profit made by Leon’s Lemonade Stand on day \( t \) of operation.

(a) What is the average profit per day between days 15 and 20?
(b) What is the average profit per day between days 15 and \( 15 + h \)?
(c) What is the instantaneous profit per day at day 15?

23. Let \( f(x) \) be a function. What is the definition of the derivative of \( f \)?

24. From the definition, find the derivative of the function \( g(x) = 5x^2 - 2x + 7 \).

25. Let \( r(x) = \frac{3x^2 - \sqrt{x} + 4}{\sqrt{x}} \). What is the instantaneous rate of change of \( r(x) \) when \( x = 2 \)?

26. Find the derivative.

\[
f(x) = 3x^6 - 8x^4 + x^2 + 52 - \frac{5}{\sqrt{x}}
\]

27. Find the derivative.

\[
g(x) = (3x^2 + 7x + 10)(x^3 - 5)
\]

28. Find the derivative.

\[
r(x) = \frac{x^2 - 3x + 2}{x^2 - x + 3}
\]

29. Find the derivative.

\[
s(x) = (2x^2 + 5x + 30)^{10}
\]

30. Find the derivative.

\[
m(x) = 8675309
\]

31. Find the derivative.

\[
n(x) = \frac{(3x + 1)^6}{(2x - 5)^4}
\]
32. Let \( r(x) = x^{1500} - 600x^3 + 3065476346 \). Find the numerical value of \( r'(1) \), and simplify it. You don’t need a calculator for this.

33. Find the tangent line to the function \( g(x) = x^3 - 5 + \sqrt{x} \) at the point \( x = 4 \).

34. (See p.187-189) Let \( C(x) = 50000 + 2000x + x^2 \) be the cost of manufacturing \( x \) scooters.
   
   (a) How much does it cost to produce 100 scooters?
   
   (b) What is the marginal cost at a production level of 100 scooters?
   
   (c) Using the marginal cost, how much would it approximately cost to produce the 101\(^{st}\) scooter?
   
   (d) What is the exact cost of producing the 101\(^{st}\) scooter?

35. Let \( I(t) \) be the income of a company in year \( t \). Interpret what it means for
   
   \[ I'(2008) = 260000. \]

36. Let \( P(t) \) be the number of hula hoops owned by children in year \( t \). Suppose you know that \( P(2005) = 100000 \) and \( P'(2005) = 5000 \). Based on the data from the year 2005, how many hula hoops would you estimate children will own in the year 2010?

37. Let \( f(x) = \sqrt{x} \).
   
   (a) Find the equation for the tangent line of \( f(x) \) at the point \( x = 9 \).
   
   (b) Using your tangent line equation, estimate the value of \( \sqrt{91} \).
   
   (c) Using a calculator, find the actual value of \( \sqrt{91} \). How does this compare with your estimate?

38. Find an antiderivative of \( g(t) = 10t^9 - 8t^5 + 3t^2 - 3t + 3 \).

39. Find
   
   \[ \int 4x + 2 - \frac{1}{x^2} \, dx. \]

40. Find all antiderivatives of \( f(t) = t^2 - 4 \).

41. Suppose that \( r'(x) = 9x^2 - 4x + 1 \) and \( r(2) = 30 \). Find \( r(x) \).

42. What function equals its own derivative?

43. Find the integral
   
   \[ \int_{-2}^{5} 3x^2 - 3x + 1 \, dx \]
44. Find the integral
\[ \int_1^1 6x^2 - 2x + 10 \, dx \]

45. Find the integral
\[ \int_1^e \frac{1}{x} \, dx \]

46. Find the integral
\[ \int_1^2 x^2 + \frac{1}{x^2} \, dx \]

47. What’s wrong with this integral?
\[ \int_{-1}^{1} 3x + \frac{1}{x} \, dx \]

48. Suppose that Leon’s Lemonade Stand has an income rate of \(4t - 10\) dollars per day on day \(t\). What is the total income Leon makes for the four days starting on day \(t = 14\)?

49. A population of snorks grows at a rate of \(2t^2 + 100t + 50\) snorks per year in year \(t\). Suppose the initial population of snorks (at time \(t = 0\)) is 1000. How many snorks will there be in year 10? How many will there be at time \(t = b\) for some \(b\)?