Math 140 Midterm 1 Review Solutions

1. The resulting graph is shifted right by 2, reflected across the x-axis, and then raised by 1.

2. Remember when sketching that on the different pieces of the domain, the different rules take effect. So when \( x < -2 \), the graph is a horizontal line, when \( -2 \leq x \leq 1 \), the graph is a parabola, and when \( x > 1 \), the graph is a straight line.

3. 1.1.7 Function
   1.1.8 Function
   1.1.9 Not a function
   1.1.10 Not a function
   1.1.11 Function
   1.1.12 Not a function

   In all the cases, the functions pass the vertical line test. The graphs which aren’t those of functions share that at some place on the graph, there is a vertical line which intersects with the graph in more than one place, hence these aren’t functions.

4. Slope:
   \[ m = \frac{-5 - 4}{7 - 3} = \frac{-9}{4} \]

   Point:
   \[ (3, 4) \]

   Point-slope formula:
   \[ y - 4 = \frac{9}{4}(x - 3) \]
   \[ y = -\frac{9}{4}x + \frac{27}{4} + 4 \]
   \[ y = -\frac{9}{4}x + \frac{43}{4} \]

5. (a) Two points: \((10, 100)\) and \((20, 200)\). Slope: \( m = \frac{200 - 100}{20 - 10} = 10 \).
   Point-slope gives \( y - 100 = 10(x - 10) \), which leads to \( y = 10x \). Thus \( f(x) = 10x \) is our function.

   (b) \( f(1) = 10 \).

   (c) \( g(x) = 2f(x) \), so \( g(x) = 20x \).

   (d) \( h(x) = 3f(x) + 20000 \), so \( h(x) = 30x + 20000 \).

   (e) No.
6. (a) Set $h(x) = 0$:

\[
x^2 - 6x + 10 = 0
\]
\[
(x^2 - 6x + 9) + 10 - 9 = 0
\]
\[
(x - 3)^2 + 1 = 0
\]
\[
(x - 3)^2 = -1
\]
\[
x - 3 = \pm i
\]
\[
x = 3 \pm i
\]

Since we have complex roots, this means that the function doesn’t cross the $x$-axis.

(b) $h(0) = 10$.

(c) In the completion of the square above, we see that the $x$-coordinate of the vertex is at $x = 3$, and the $y$-coordinate is at $y = 1$, so the vertex is at $(3, 1)$.

(d) Take the standard parabola $y = x^2$, and shift it three to the right and one up.

(e) For a general parabola, the graph crosses the $x$-axis exactly where the roots are. In the quadratic formula, the discriminant is $b^2 - 4ac$, and since we are taking the square root of this, its sign will tell us information about the roots. If $b^2 - 4ac > 0$, then it has a positive square root, and there will be two real distinct solutions. If $b^2 - 4ac = 0$, then there will be a single real root, but it will have a double factor. If $b^2 - 4ac < 0$, then its square root is imaginary, which means we have only complex solutions, and in this case the graph will not cross the $x$-axis.

7. For example 2.1.7:

(a) There are three turning points on the graph, i.e. three spots where the graph is at a local maximum (where the graph turns back downwards) or at a local minimum (where the graph turns back upwards).

(b) The maximum number of turning points is one less than the degree, so this means that the minimum of the degree has to be one more than the number of turning points. In this example, the degree has to be at least 4.

(c) Since the graph starts down and heads back down in the long term, the leading coefficient of this polynomial is negative.

8. The highest power which appears in the polynomial is 1001, so the degree of this polynomial is 1001. The Fundamental Theorem of Algebra tells us that if we count up all the roots with multiplicities, then the number of roots is equal to the degree. Thus this polynomial has at most 1001 distinct roots.
9. (a) \( r(x) = 0 \) leads to \( x = \frac{3}{2} \).
(b) \( r(0) = -3 \).
(c) Doing the division gives \( r(x) = 2 \cdot \frac{5}{x+1} \), so the horizontal asymptote is \( y = 2 \), and the vertical asymptote is \( x = -1 \).
(d) Using the asymptotes and the intercepts, this graph can be seen to look similar to the graph of \( y = \frac{1}{x} \), except our function \( r(x) \) is shifted, reflected vertically, and shifted vertically.

10.
\[
3^{5x+5} = (3^2)^{3x+1} \\
3^{5x+5} = 3^{2(3x+1)} \\
3^{5x+5} = 3^{6x+2} \\
5x + 5 = 6x + 2 \\
x = 3
\]

11.
\[
A = (1500) \left(1 + \frac{12}{4}\right)^{4^{10}} \\
A = 1500(1.03)^{40}
\]

12.
\[
2 \log_5 x + \log_5 7 - 3 \log_5 y + 1 = \log_5 x^2 + \log_5 7 + \log_5 y^{-3} + \log_5 5 \\
= \log_5 (x^2 \cdot 7 \cdot y^{-3} \cdot 5) \\
= \log_5 \left(\frac{35x^2}{y^3}\right)
\]

13. (a) \( \log_3 9 = 2 \)
(b) \( \log_4 32 = \log_4 2^5 = 5 \log_4 2 = 5 \left(\frac{1}{2}\right) = \frac{5}{2} \)
(c) \( \log_5 5^{29} = 29 \)
(d) \( 6^{\log_6 7} = 7 \)

14. Solve the equations
(a)

\[4^{3x-1} = 50\]
\[\ln 4^{3x-1} = \ln 50\]
\[(3x - 1) \ln 4 = \ln 50\]
\[3x \ln 4 - \ln 4 = \ln 50\]
\[3x \ln 4 = \ln 50 + \ln 4\]
\[x = \frac{\ln 50 + \ln 4}{3 \ln 4}\]

(b)

\[3^{x+1} = 5^x\]
\[\ln 3^{x+1} = \ln 5^x\]
\[(x + 1) \ln 3 = x \ln 5\]
\[x \ln 3 - x \ln 5 = -\ln 3\]
\[x(\ln 3 - \ln 5) = -\ln 3\]
\[x = \frac{-\ln 3}{\ln 3 - \ln 5}\]

(c)

\[\log_5(x - 1) + \log_5(x + 3) = 1\]
\[\log_5 [(x - 1)(x + 3)] = 1\]
\[5^{\log_5 [(x - 1)(x + 3)]} = 5^1\]
\[(x - 1)(x + 3) = 5\]
\[x^2 + 2x - 3 = 5\]
\[x^2 + 2x - 8 = 0\]
\[(x + 4)(x - 2) = 0\]

So either \(x = -4\) or \(x = 2\). There’s a problem if \(x = -4\), though; we would be taking logarithms of negative numbers, which is a problem. If \(x = 2\) we don’t have this problem. Thus we only have one solution of this equation, and it is \(x = 2\).
15. 

\[9000 = 2000 \left(1 + \frac{12}{12}\right)^{12t}\]

\[\frac{9}{2} = (1.01)^{12t}\]

\[\ln \frac{9}{2} = \ln 1.01^{12t}\]

\[\frac{\ln \frac{9}{2}}{12 \ln 1.01} = t\]

16. Since we are collecting compound interest, we will use the formula \(A = Pe^{rt}\). In this case, \(P = 1500\), \(A = 15000\), and \(t = 20\). Putting these into the equation:

\[15000 = 1500e^{-20}\]

\[10 = e^{-20t}\]

\[\ln 10 = \ln e^{-20t}\]

\[\ln 10 = -20t\]

\[\frac{\ln 10}{-20} = t\]

17. Using the properties of limits, find

\[\lim_{x \to 3} \sqrt{4x + 4} = \sqrt{\lim_{x \to 3} (4x + 4)}\]

\[= \sqrt{(\lim_{x \to 3} 4) + \lim_{x \to 3} 4}\]

\[= \sqrt{4 \cdot 3 + 4} = 4\]

18. Consider the function

\[f(x) = \frac{x^2 - 9}{x - 3}\]

(a) Using the properties of limits, find \(\lim_{x \to 3} f(x)\), if it exists.

Factoring the numerator, we find \(f(x) = \frac{(x+3)(x-3)}{x-3} = x+3\), provided that \(x \neq -3\). Since we are asking about the limit as \(x\) approaches 3 and not actually at the point itself, \(\lim_{x \to 3} (x+3) = 3 + 3 = 6\). We can plug in \(x = 3\) in this case, since our function \(f(x)\) is continuous at \(x = 3\).

(b) Find \(f(3)\), if it exists.

\(f(3) = 6\).

(c) Does \(\lim_{x \to 3} f(x) = f(3)\)? Yes.
If we ask the same questions at or near \( x = -3 \), then we see that the function is not defined there, but the limit does exist, and we can define \( f(-3) \) so the function is continuous (in fact the function would become just \( f(x) = x + 3 \)).


(a) For which \( x \) values in the domain is this function discontinuous?
\( x = -2, x = 1, \) and \( x = 2 \).

(b) For these \( x \) values you found, for which ones can you define or redefine \( f(x) \) at that point so that the function is continuous.
\( x = -2 \) and \( x = 2 \), since the limit of \( f(x) \) at these points exists.
There is no way to redefine \( f(1) \) to make the function continuous there since the limit doesn’t exist at \( x \to 1 \).

(c) For which points \( x = c \) will the function always be discontinuous, regardless of the value of \( f(c) \)?
\( x = 1 \) for the reason above.

20. Sketch a graph of a function \( y = g(x) \) which has these properties:

(a) \( \lim_{x \to -2^-} g(x) = 1 \)

(b) \( \lim_{x \to -2^+} g(x) = 3 \).

(c) \( g(2) = -1 \).

(d) \( g(x) \) is continuous for all \( x \neq 2 \).

(e) \( g(-2) = 4 \).

(Note: There are a lot of possible solutions to this question.)
You need to sketch a graph with \( g(-2) = 4 \); as \( x \) gets bigger and heads towards 2, the function value should head toward 1. At the value \( x = 2 \), itself, there is a discontinuity as \( g(2) = -1 \), which doesn’t match with the values of \( g \) as \( x \) approaches 2 from below. Now, as \( x \) leaves 2 and gets larger, the \( y \) value of the graph seemingly departs as if from 3 (that’s where the \( \lim_{x \to -2^+} g(x) = 3 \) comes in) then continues continuously.