A Special Case Solution to the Perspective 3-Point Problem
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Abstract

In this paper we address a special case of the perspective 3-point problem. We assume that the 3 points form an equilateral triangle, and that the center of perspective lies on a plane perpendicular to the triangle and containing one of its altitudes. We then present a geometric proof that accounts for all possible solution sets, from 1 to 4 solutions. We emphasize the shared-point and shared-side solution pairs, as first developed in reference [5].

Introduction

Given the perspective view of 3 non-colinear control points (i.e.: known ΔABC) we wish to determine the possible camera-triangle configurations. This problem, called the perspective 3-point problem (P3P), has been addressed by several authors. Reference [1] shows that there can be at most 4 solutions, and a specific 4-solution case is constructed. Reference [2] shows that there are exactly 4 solutions whenever the center of perspective (CP) is directly over the orthocenter of the triangle. Reference [3] delineates the history of the problem and discusses the relative effectiveness of the various algebraic solutions that have been proposed in the literature. Reference [4] provides a complete algebraic solution along with some geometric techniques to help visualize configurations with 1, 2, 3 or 4 solutions. Reference [5] focuses on the special case where there is at least one solution with CP on a plane perpendicular to the triangle and containing one of the its altitudes, and shows that there are 4 solutions, provided that CP is within the "danger cylinder". Reference [6] identifies the "danger cylinder" as the region of instability caused by the degeneracy of the Jacobian of the nonlinear equations that define the camera-triangle constraints. Reference [7] suggests that the instability is related to the existence of exactly 3 solutions on the boundary of the danger cylinder.

Here we address the same special case as developed in [5] but we add an additional assumption: the known triangle is an equilateral triangle. Then, as should be expected, we can push the interesting results from [5] even further. In particular, a "Key Theorem" facilitates the derivation of a relatively straightforward geometric solution. To our knowledge, this is the most complex case for which a complete geometric characterization of all solutions is unambiguously provided.

The Special Case

We assume a pin-hole camera model and that the correspondence between the image points and the corners of the triangle (A, B, C) is known. Figure 1a shows the notation
for the assumed equilateral triangle. As in [5], we assume that there is at least one solution with CP on a plane orthogonal to the triangle and containing one of its altitudes, as shown in figure 1b. Without any further loss of generality let side $AB$ be the base of that altitude, and label the orthogonal plane $\pi_{AB}$. We also know that $CPm$ is a perpendicular bisector of $AB$, and that $CPm$ bisects $\theta = \angle ACPB$.

Figure 1. The triangle is equilateral with side $s$ and altitude $h$. $O$ is the orthocenter of the triangle, and $m$ is the midpoint of $AB$. $CP$ is on the plane $\pi_{AB}$, which is perpendicular to the plane of the triangle and contains the altitude $Cm$.

**Approach**

We start with the triangle in any of the configurations that satisfy our assumptions, such as the configuration shown in figure 1b. To get multiple solutions we maneuver the triangle into other configurations that generate the same perspective measurements. From our assumptions it is clear that the camera's perspective measurements can be used to calculate $\theta$ (\angle ACPB) and $\phi$ (\angle CCPm), as labeled in figure 1b (and in figure 2). We think of $\phi$ as the "angle of elevation" of the line $CPC$ above $CPm$. Because of our assumptions the problem reduces to 2 rather than 3 measured angles. Generally, the perspective measurements consist of the 3 angles subtended by each of the three sides of the triangle, but in our special case the problem reduces to the measurement of $\theta$, the angle subtended by side $AB$, and $\phi$, the angle elevation of $C$.

We will use the geometry of the initial configuration to generate all possible camera-triangle configurations that are consistent with the measured values of $\theta$ and $\phi$. In each
case we will provide geometric constructs and explicit formulas for the locations of the triangle with respect to the camera.

Shared-Side Solution Pair

Given the initial configuration (figure 1b), there is an easy way to generate a second solution. Rotate about $\overline{AB}$ until $C$ hits the same line of perspective again. This creates a "shared-side solution pair" similar to those that are defined in [5] (see figure 2). This pair of solutions shares side $\overline{AB}$. We sometimes refer to this pair as the "front-back pair" because one solution is typically leaning frontward (toward CP) while the other is leaning backward (away from CP). There are a couple of exceptions. If the starting solution is such that the triangle's altitude ($\overline{Cm}$) is perpendicular to the line of perspective running through $C$ ($\overline{CPC}$), then rotating about $\overline{AB}$ will not produce another solution (i.e. because of tangency). In this case the shared-side "pair" reduces to a single solution. This point of tangency also identifies the largest value of $\phi$ that is feasible for the given $\theta$:

$$\phi_{\text{max}} = \arcsin(\sqrt{3} \tan \frac{\theta}{2})$$

The other exception is when the new intersection is behind the camera (i.e.: when the distance from CP to m is less than h). These situations are easy to diagnose and deal with.

![Diagram](image)

Figure 2: The "shared-side" solution pair. By rotating about $\overline{AB}$ the point C can meet the line of perspective in 2 places. Both configurations have the same perspective measurements. For our special case, the perspective measurements reduce to the measurement of angles $\theta$ and $\phi$.  

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Shared-Point Solution Pair

Starting with the initial configuration (figure 1b), while keeping $\theta$ fixed, slide $\overline{AB}$ toward CP. Label the new location of the triangle $A'B'C'$ (see figure 3). If $A'$ is closer to CP (as it is in figure 3) then there is a symmetric location with $B'$ closer to CP. This symmetric pair will form the "shared-point solution pair" as also described in [5]. We sometimes refer to this pair as the "left-right" pair. After moving $\overline{AB}$ to one of these locations we can rotate about $\overline{A'B'}$ until $C'$ intersects $\pi_{AB}$ (see figure 4). Now, we prove the Key Theorem.

Figure 3: While keeping $\theta$ fixed, $\triangle ABC$ has been slid to a new location, labeled $\triangle A'B'C'$. Then, the triangle is rotated about $\overline{A'B'}$ to move $C'$ into $\pi_{AB}$.

Figure 4: The triangle can be slid, keeping $\theta$ fixed, to many possible locations, but in each case we can rotate about $\overline{A'B'}$ until $C'$ intersects $\pi_{AB}$. On the left side of the figure the triangle has been slid as close to CP as possible ($A'$, or $B'$, coincident with CP). On the right side of the figure the triangle has been slid as far away from CP as possible, in which case the left and right handed symmetric pair reduces to a single triangle that leans back at the same angle as any of the other configurations shown in the figure. The Key Theorem proves that the height of the intersection ($z$), as well as the leaning angle, is the same for each of these configurations.
The height at which C' intersects the orthogonal plane in a shared-point pair, as shown in figures 3 and 4, is given by:

\[ z = h\sqrt{1 - \frac{1}{3} \tan^2 \frac{\theta}{2}} \]

Proof: Refer to the construction in figure 5. Using a sequence of similar right triangles we have:

\[ \Delta A_R p_1 B_R \approx \Delta p_3 p_2 B_R \]
\[ \frac{(L_2 - L_1) \cos(\frac{\theta}{2})}{t} = \frac{(L_2 + L_1) \sin(\frac{\theta}{2})}{L_1 \sin(\frac{\theta}{2})} \]
\[ \Rightarrow t = \frac{(L_2 - L_1)L_1 \cos(\frac{\theta}{2})}{(L_2 + L_1)} \]

\[ \Delta p_3 p_2 B_R \approx \Delta p_3 p_2 B_L \]
\[ \frac{\frac{\theta}{2} + x}{\frac{\theta}{2} - x} = \frac{L_2 \sin(\frac{\theta}{2})}{L_1 \sin(\frac{\theta}{2})} \]
\[ \Rightarrow x = \frac{s(L_2 - L_1)}{2(L_2 + L_1)} \]

Using one more pair of similar right triangles, and substituting the results for \( t \) and \( x \), we get:

\[ \Delta p_3 p_4 m_L \approx \Delta p_3 p_2 B_R \]
\[ \frac{y}{L_1 \sin(\frac{\theta}{2})} = \frac{x}{t} \]
\[ \Rightarrow y = \frac{1}{2} \tan(\frac{\theta}{2}) \]

Now, \( y \) and \( z \) are the legs of a right triangle with hypotenuse \( h \), as shown in figure 3. Therefore:

\[ z = h\sqrt{1 - \frac{1}{3} \tan^2 \frac{\theta}{2}} \]

QED.
Figure 5: Looking down on the plane formed by A, B, and CP. The side $\overline{AB}$ has been slid, keeping $\theta$ fixed, to an arbitrary position. For any such position there is a symmetric position. The figure labels one of them $\overline{A_L B_L}$ and the other $\overline{A_R B_R}$ (L for "left" and R for "right"). The Key Theorem shows that the variable $y$ depends only on $\theta$, independent of $L_1$ and $L_2$.

Note that the only property of the triangle that is used in the proof of the theorem is that $\overline{CPm}$ is a perpendicular bisector of $\overline{AB}$. Therefore the theorem is also true if the triangle is isosceles (with base side $\overline{AB}$).
It is surprising that $z$ depends only on $\theta$, and not on $L_1$ and $L_2$. Since $z$ depends only on $\theta$, we can plot the locus of points in $\pi_{AB}$ at height $z$ above $CPm$. Each of these points is the potential location of a share-point solution pair. Consider the line $\lambda$, in $\pi_{AB}$, at height $z$ above $CPm$ (see figure 4). Whenever the line of perspective (from CP to C') intersects $\lambda$ we have the potential location of a shared-point solution pair.

Figure 6 looks perpendicularly at $\pi_{AB}$. The circle in the figure has radius $h$, and it is centered at $m$, the midpoint of $AB$. The side $AB$ is shown only as a single point, labeled "m" in the figure. This circle defines the locations of share-side solutions (as was already shown in figure 2). The shared-side solutions correspond to the intersections of the line of perspective and the circle. The share-point solutions correspond to the intersections of the line of perspective with $\lambda$.

Notice that the only intersections on $\lambda$ that produce feasible shared-point candidates are those between $c_3$ and $c_1$. If the intersection is to the left of $c_3$ then the configuration violates our assumptions ($\phi > \phi_{max}$). And, as was previously explained, $c_1$ is the furthest position that the lean-back solution can reach for our given configuration. Therefore, the only candidates on $\lambda$ for feasible shared-point solutions occur between $c_3$ and $c_1$. 

![Figure 6: An orthogonal view of the $\pi_{AB}$. Lines of perspective emanating from CP that intersect the circle at two places specify the locations of shared-side solution pairs. A line](image-url)
of perspective that intersects $\lambda$, anywhere between c3 and c1, is the potential location of a shared-point solution pair.

Combining Shared-Side and Shared-Point Solutions

Position 1 in figure 6 shows a side view of the triangle as it leans back at the angle:

$$\arctan \left( \frac{3}{\sqrt{\tan^2 \frac{\theta}{2} - 1}} \right)$$

This corresponds to the far right part of figure 4, where the triangle is in the position with $\overline{AB}$ perpendicular to $\overline{CP_m}$ and leaning back at the same angle as each of the shared-point configurations. That is, this is where the "left" and "right" members of the shared-point pair become a single solution. In particular, lines of perspective that intersect $\lambda$ beyond Position 1 do not correspond to alternate solutions since the triangle cannot be that far away from CP and still have $\overline{AB}$ perpendicular to $\overline{CP_m}$ and subtending the angle $\theta$. Position 1 corresponds to the smallest value of $\phi$ for which shared-point solutions are possible.

Lines of perspective emanating from CP and intersecting the circle below Position 1 correspond to configurations with exactly 2 solutions, coming from shared-side pairs. These are somewhat extreme front-back pairings, with the extreme case being $\phi = 0$ (CP in the plan of the triangle).

As the line of perspective rises (i.e.: $\phi$ increases) from Position 1 to Position 2 we have both shared-side and shared-point solution pairs, as indicated by the intersection twice with the circle and once with $\lambda$. This results in configurations with exactly 4 solutions.

Position 2 is where CP is directly over the orthocenter of the triangle. Notice its symmetric positioning with respect to Position 1 (a consequence of the equilateral triangle assumption). At Position 2, each of the shared-point solutions share a common side with one of the shared-side solutions (the "lean-back" solution), which is consistent with the solutions derived in [2].

Between Position 2 and Position 3 we still have 4 solutions coming from distinct shared-side and shared-point pairs, but the shared-side solutions get closer and closer as we approach Position 3.

At Position 3 the line of perspective is tangent to the circle ($\phi = \phi_{\text{max}}$), in which case the shared-side configuration becomes a single solution, but the shared-point solutions are still present as is indicated by the intersection of the line of perspective with $\lambda$ at c3. Therefore, there are 3 solutions at Position 3.

Given the angles $\phi$ and $\theta$, the exact positions of each triangle in solutions described above can be derived in a straightforward manner from the equations in the Key Theorem and the labels in figures 5 and 6.
Varying θ

That accounts for all the feasible solutions when CP is at least a distance h from m, as it is in figure 6. If CP gets closer to the triangle, the same type of sketch shows that there are situations with exactly 1 solution because the "front" member of the shared-side solution is now behind the camera. However, for large enough value of φ, it is possible to have shared-point solutions, which in combination with the single shared-side solution produces exactly 3 solutions. For a systematic look at such possibilities, we let θ vary.

Let \( d = |\overline{CPm}| \) be the distance from CP to m. From figure 5 we see that d is inversely related to θ:

\[
d = \frac{h}{\sqrt{3} \tan \frac{\theta}{2}}
\]

Let \( \phi_1, \phi_2 \) and \( \phi_3 \) be the elevation angles of Position 1, Position 2 and Position 3 respectively, in figure 6. Given any value of d, we can compute the values of \( \phi_1, \phi_2 \) and \( \phi_3 \) (from figure 6) as functions of θ:

\[
\phi_1 = \arctan\left(\frac{T \sqrt{3} - T^2}{1 + T^2}\right)
\]

\[
\phi_2 = \arctan\left(\frac{T \sqrt{3} - T^2}{1 - T^2}\right)
\]

\[
\phi_3 = \arctan\left(\frac{T \sqrt{3}}{\sqrt{1 - 3T^2}}\right)
\]

\[
T = \tan \frac{\theta}{2}
\]

From these equations we can plot \( \phi_1, \phi_2 \) and \( \phi_3 \) as a function of the distance d, as shown in figure 7. If \( \phi \), the angle of elevation of the line of perspective, is less than or equal to \( \phi_1 \) then we will have exactly 2 solutions, the shared-side pair. If \( \phi_1 < \phi < \phi_3 \) there are exactly 4 solutions, the combined shared-point and shared-side pairs. When \( \phi = \phi_2 \), CP is directly over the orthocenter. \( \phi \) cannot be greater than \( \phi_3 \) (= \( \phi_{\text{max}} \)).

As d increases, θ decreases gradually to 0 while Positions 1, 2 and 3 in figure 6 move toward the top of the circle (converging to a single point at the top of the circle when \( d = \infty \), \( \theta = 0 \)).

When \( d = h \), the point of tangency (Position 3) is coincident with CP and we have \( \phi_3 = \pi/2 \) and \( \theta = \pi/3 \).
Figures 7 and 8 show the possible solution sets for \( d \leq h \). As \( d \) decreases, Positions 1, 2, and 3 move down the circle. When \( d = h \), CP is coincident with the point of tangency (Position 3) and \( \phi_3 = \pi/2, \theta = \pi/3 \). When \( d = \frac{h}{\sqrt{3}} \), CP can no longer be directly over the orthocenter because \( \phi_2 = \pi/2 \). At \( d = h/3 \) the shared-point solutions are no longer geometrically possible (\( \phi_1 = 0, z = 0, \theta = 2\pi/3 \)). See Chart 1 for a summary of combinations of \( d, \theta \) and \( \phi \).

Figure 8 shows the same graph a figure 7, but this time with labels indicating the exact number of solutions associated with each configuration. Figure 8 omits \( \phi_2 \) since it simply tells us when CP is directly over the orthocenter, and does not cause a change in the number of solutions.

<table>
<thead>
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<th>( d )</th>
<th>( \theta )</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
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<td>( \tan^{-1}(\sqrt{2}/2) )</td>
<td>( \tan^{-1}(\sqrt{2}) )</td>
<td>( \pi/2 )</td>
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<tr>
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<td>( 2\pi/3 )</td>
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<tr>
<td>0</td>
<td>( \pi )</td>
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Figure 7: When CP is at a distance \( h \) from \( m \), Position 3 (from figure 6) is coincident with CP. At this distance we lose one of the shared-side solutions because the lean-forward solution is behind the camera. At \( d = \frac{h}{\sqrt{3}} \) the CP can no longer be over the orthocenter (\( \phi_2 = \pi/2 \)). At \( d = h/3 \) the shared-point solutions are lying flat (\( \phi_1 = 0, z = 0, \theta = 2\pi/3 \)), and if \( d < h/3 \) (\( \theta > 2\pi/3 \)) the shared-point solutions are no longer possible.
Chart 1: Values of $\theta$, $\phi_1$, $\phi_2$ and $\phi_3$ corresponding to sampled values of $d$ between $h$ and 0. The *'s indicate infeasible configurations. For example, when $d = h$, $\phi_3 = \pi/2$, but when $d < h$ Position 3 goes behind the camera, so $\phi_3$ is ignored.

Figure 8: Same as figure 7, but with $\phi_2$ omitted, and labeled with the number of solutions associated with each region of values for $\phi$ and $d$. When $d$ is between 0 to $h/3$ we get a single solution: the lean-back member of the shared-side pair. No other solution is geometrically possible. When the distance is between $h/3$ and 1 there are two possibilities. If $\phi \leq \phi_1$, then we again have only one solution, the lean-back member of the shared-side pair. If $\phi > \phi_1$ then the shared-point solutions are possible and together with the lean-back solution make 3 distinct solutions. For $d > h$, we have two possibilities. If $\phi \leq \phi_1$ we get exactly 2 solutions, the shared side pair. If $\phi > \phi_1$ we get exactly 4 solutions up until $\phi = \phi_3$, which is where the tangency occurs, so we lose one of the shared-side solutions, resulting in exactly 3 solutions.

**Fixed Triangle**

Figure 9 shows the same configurations as figure 6 but this time with the triangle fixed and CP moving along a circle in the orthogonal plane. This circle has radius $r = \frac{s}{2 \tan \frac{\theta}{2}}$, and it is centered at $m$. Included in the picture is the location of the "danger cylinder", the right cylinder whose base is the circle that circumscribes the triangle. Positions 1 and 3 correspond to being on the boundary of the danger cylinder, and Position 2 corresponds to being directly over the triangle's orthocenter.
Figure 9: The same configurations as figure 6, but with a fixed triangle, and including the location of the danger cylinder. Positions 1 and 3 are on the boundary of the cylinder and Position 2 is directly over the orthocenter. Position 3 is clearly the place where the triangle's altitude is tangent to the line of perspective. Shared-point solutions are possible when CP is between Position 1 and Position 3, as are the shared-side solutions, for a total of 4 solutions. It is easy to see the symmetry between Positions 1 and 2 since they are located directly above points at distance $h/3$ from $m$, which is how we know that Position 1 is on the boundary of the cylinder. If CP is on the same circular trajectory but outside the cylinder, then there are exactly 2 solutions, the front-back pair.

**Conclusions**

For the special case of an equilateral triangle, and the center of perspective in one of the orthogonal planes, we have provided a thorough geometric solution. Some of the analysis can be directly extended to the case of an isosceles triangle, but then the results would be restricted to the specific orthogonal plane that is orthogonal to the base of the triangle, whereas the results presented here apply to any of the three orthogonal planes.

Another extension of these results would be to consider a small perturbation off the orthogonal plane. If CP were close to, but not on, the orthogonal plane then the shared-side solutions would be eliminated. That is, the shared-side solutions depend heavily on the symmetry created by the orthogonal plane. Once that symmetry is gone, it appears that the only possible solutions are those that are small adjustments to the share-point solutions, which would no longer share a point but would be close to their configurations as in the orthogonal case. This leads us to the conjecture that 2-solution case is the norm.
and that the 4-solution case results from geometrically rare (set of measure 0) symmetries.

References


