Math 300: Homework Set #7.

1. Find generating functions for each of the following sequences (assume sequences have indices beginning with 0). If a generating function has infinitely many nonzero terms, express it in closed form (via geometric series, etc.).

   (a) 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, ...
   (b) 0, 0, 0, 1, 1, 1, 1, 1, ...
   (c) 0, 1, 0, 0, 1, 0, 0, 1, ...
   (d) 2, 4, 8, 16, 32, 64, ...
   (e) 1, \( \binom{7}{0} \), \( \binom{7}{1} \), ..., \( \binom{7}{7} \), 0, 0, 0, 0, ...

2. Use generating functions to determine the number of different ways 10 identical balloons can be given to 4 children if each child receives at least 2 balloons.

   (a) Explain why the generating function is \( G(x) = (x^2 + x^3 + ...)^4 \).
   (b) Why can we rewrite this as \( G(x) = x^8(1 - x)^{-4} \)? Use the negative binomial series and find the coefficient of \( x^{10} \) to solve this exercise.

3. Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to 6 children if each child receives at least 1 but no more than 3 stuffed animals.

4. Use generating functions to find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

5. What is the generating function for \( \{a_k\} \), where \( a_k \) is the number of solutions of \( w + x + y + z = k \), where \( w, x, y, z \) are integers with \( w \geq 3, 1 \leq x \leq 5, 0 \leq y \leq 4, \) and \( z \geq 1 \)? Use this answer to find \( a_7 \).

6. Use generating functions to find the number of ways to make change for 100 dollars using $10, $20, and $50 bills.

7. If \( G(x) \) is the generating function for the sequence \( \{a_k\} \), what is the generating function (in terms of \( G(x) \)) for each of the following sequences?

   (a) \( 2a_0, 2a_1, 2a_2, 2a_3, ... \)
   (b) \( 0, a_0, a_1, a_2, ... \)
   (c) \( a_2, a_3, a_4, ... \)
   (d) \( a_1, 2a_2, 3a_3, 4a_4, ... \) (Calculus is required.)
   (e) \( 0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, ... \) (Calculus is required.)
   (f) \( a_0, a_0 + a_1, a_0 + a_1 + a_2, ... \) (Consider the geometric series with \( G(x) \).)