1. (p. 216, #3 from the text) In a class of 50 students, the number of students of ages 18, 19, 20, 21, 25 are 20, 22, 4, 3, 1, respectively. Let $X$ be the random variable, where $X$ is one of the five ages above. First, find the probability mass function for $X$. Then, compute its expected value (mean) and variance.

2. (p. 233, #9 from the text) Let $X$ have the discrete uniform distribution on the integers 1, 2, ..., $n$. (This means that $P(X = n) = \frac{1}{n}$; that is, all $n$ integers are equally likely.) Compute the variance of $X$. The formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$ will be useful.

3. (p. 233, #4, 5 from the text) Let $X$ be a (discrete, though it does not truly matter) random variable for which $E[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

   (a) Show that $E[X(X - 1)] = \mu(\mu - 1) + \sigma^2$.

   (b) Show that $E[(X - c)^2] = (\mu - c)^2 + \sigma^2$ for any constant $c$.

4. (a) Recalling the geometric series $\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k$, differentiate both sides and then multiply both sides by $x$ to show that

   $$\frac{x}{(1 - x)^2} = \sum_{k=1}^{\infty} kx^k.$$

   (b) Taking the result from part (a), differentiate both sides and then multiply both sides by $x$ to deduce that

   $$\frac{x(1 + x)}{(1 - x)^3} = \sum_{k=1}^{\infty} k^2x^k.$$

   (c) Deduce that if $X$ is a random variable for the geometric distribution with parameters $p$ and $q = 1 - p$, then $E[X^2] = \frac{1 + q}{p^2}$. Conclude that $\text{Var}(X) = \frac{q}{p^2}$.

5. If $X$ is a random variable for the Poisson distribution with parameter $\lambda$, then $E[X^2] = \lambda(\lambda + 1)$. Conclude that $\text{Var}(X) = \lambda$.

6. Suppose that $Y$ is a continuous random variable which has probability density function $f(y) = cy^2(1 - y)$ for $y \in [0, 1]$ and 0 otherwise. First, find $c$ so that $f(y)$ is a pdf. Then, compute $E[Y^2]$. 

Math 352, Homework Set #5.
7. **Bonus:** (Mean and Variance for a Negative Binomial Distribution.)
For this exercise, let $X$ denote the number of failures which occur before the $r$-th success. Then, we can write the probability mass function for the Negative Binomial Distribution as

$$P(X = k) = \binom{r + k - 1}{k} p^r q^k, \text{ where } k = 0, 1, 2, \ldots.$$  

(a) Verify that

$$\sum_{k=0}^{\infty} P(X = k) = 1.$$  

*Hint:* Use the ‘negative’ binomial series $(1 - x)^{-r} = \sum_{k=0}^{\infty} \binom{r + k - 1}{k} x^k$.

(b) Show that $E[X] = \frac{rq}{p}$ and $\text{Var}(X) = \frac{rq}{p^2}$. 