Math 511, The Tenth Homework Set.

Please answer at least 6 of the 7 problems below for full credit.

1. Show that a sublinear functional \( p \) satisfies \( p(0) = 0 \) and \( p(-x) \geq -p(x) \).

2. To illustrate Theorem 4.3-3 (Bounded linear functionals), let \( X = \mathbb{R}^2 \) and find a functional \( \tilde{f} \) as asserted in the conclusion to this theorem.

3. Show that under the assumptions of Theorem 4.3-3, there is a bounded linear functional \( \tilde{f} \) on \( X \) such that \( \| \tilde{f} \| = \| x_0 \|^{-1} \) and \( \tilde{f}(x_0) = 1 \).

4. Show that \( T : \mathbb{R}^2 \to \mathbb{R} \) defined by \( T(x, y) = x \) is an open mapping. Is \( T' : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T'(x, y) = (x, 0) \) an open mapping?

5. Show that an open mapping need not map closed sets to closed sets.

6. Let \( X \) be a normed space whose points are sequences of complex numbers \( x = (\xi_j) \) with only finitely many nonzero terms and norm defined by \( \| x \| = \sup_{j \in \mathbb{N}} |\xi_j| \). Let \( T : X \to X \) be defined by \( T x = (\xi_1, \frac{1}{2} \xi_2, \frac{1}{3} \xi_3, ...) \). Show that \( T \) is linear and bounded, but \( T^{-1} \) is unbounded. Why does this not contradict the Open Mapping Theorem?

7. Show that the graph \( G(T) \) of a linear operator \( T : X \to Y \) is a vector subspace of \( X \times Y \).