Math 511, The Seventh Homework Set.

**Note:** This assignment covers sections 3.1 and 3.2 from the textbook.

1. Fix constants $w_1, ..., w_n > 0$ and define for $x, y \in \mathbb{R}^n$:
   \[
   \langle x, y \rangle = \sum_{k=1}^{n} w_k x_k y_k.
   \]
   Verify that this yields an inner product on $\mathbb{R}^n$. How would we need to modify this definition for it to yield an inner product on $\mathbb{C}^n$? What about $l^2(\mathbb{R})$?

2. (Pythagorean Theorem) Show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ for any orthogonal vectors $x$ and $y$ in an inner product space $X$. Extend the formula to $m$ mutually orthogonal vectors.

3. If $X$ is a real inner product space, show that the condition $\|x\| = \|y\|$ implies that $\langle x + y, x - y \rangle = 0$. What does this mean geometrically if $X = \mathbb{R}^2$? What does this condition imply if $X$ is complex?

4. (a) If $V$ is a real inner product space, then show $\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)$.
   (b) If $V$ is a complex inner product space, then show $\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2 + i \|u + iv\|^2 - i \|u - iv\|^2)$.

5. If $y$ is orthogonal to $x_n$ for all $n \in \mathbb{N}$, and $\{x_n\} \to x$, show that $x$ is orthogonal to $y$.

6. Show that for a sequence $\{x_n\}$ in an inner product space, the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ imply the convergence $\{x_n\} \to x$.

7. Show that in an inner product space, $x$ is orthogonal to $y$ iff $\|x + \alpha y\| \geq \|x\|$ for all scalars $\alpha$.

8. Suppose that $X$ is a finite dimensional inner product space with orthogonal basis $\{e_1, ..., e_n\}$. Prove that $x = \sum_{k=1}^{n} c_k e_k$, where $c_k = \frac{\langle x, e_k \rangle}{\langle e_k, e_k \rangle}$.