Math 582, The Second Homework Set.

For next class, skim through the remainder of Chapter 2 as well as Chapter 3 in the text by Stewart and Tall.

1. Fix an integer $n$ which is not a perfect square and define the set

$$\mathbb{Z}[\sqrt{n}] = \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\}.$$

(a) Show that $\mathbb{Z}[\sqrt{n}]$ is closed under addition, additive inverses, and multiplication. (Being a subset of the ring $\mathbb{C}$, this shows that $\mathbb{Z}[\sqrt{n}]$ is a ring.)

(b) Give a condition which will determine whether an element in $\mathbb{Z}[\sqrt{n}]$ has a multiplicative inverse.

2. Let $M = \left\{ \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix} : a \in \mathbb{Z} \right\}$. I'm telling you that this is a commutative ring with respect to matrix addition and multiplication.

Letting $a \in \mathbb{Z}$, consider the map $\phi : \mathbb{Z} \to M$ defined by $\phi(a) = \begin{pmatrix} 0 & 0 \\ a & a \end{pmatrix}$.

(a) Show for $\phi$ is a ring homomorphism:

For any $a, b \in \mathbb{Z}$, $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$.

(b) Show that $\phi$ is onto.

(c) Show that $\ker \phi = \left\{ a \in \mathbb{Z} \mid \phi(a) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \{0\}$. Is $\phi$ a 1-1 map?

3. In $\mathbb{Z}$, define the ideals $I = \langle 6 \rangle$ and $J = \langle 8 \rangle$.

(a) What are the elements in $I$?

(b) Compute $I \cap J$ and $I + J$. Simplify your answers as far as possible. What do you notice?

4. Show that the following numbers are algebraic integers. Can you find their minimal polynomials?

(a) $4$

(b) $\sqrt{7}$

(c) $4 - 3i$

(d) $\frac{3 + \sqrt{-3}}{2}$

(e) $\sqrt{2} + \sqrt{5}$.

5. Given the set of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, try to factor all the prime numbers of $\mathbb{N}$ up to 50 in $\mathbb{Z}[i]$. Some will factor (like 5 = $(2 + i)(2 - i)$), while others will not (like 3). Do you notice any pattern?