Math 582, The Eighth Homework Set.

1. Use Dedekind’s Theorem to factorise the following principal ideals in the ring of integers of the following fields. (Note that these rings are UFDs.)

   (a) \( \mathbb{Q}(\sqrt{3}) \): \( \langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 10 \rangle, \langle 30 \rangle \).

   (b) \( \mathbb{Q}(\sqrt{2}) \): \( \langle 7 \rangle, \langle 29 \rangle, \langle 31 \rangle \).

2. We’ll look at how rational primes behave in the number ring \( \mathbb{Z}[\sqrt{2}] \). I’m telling you that this is a UFD.

   (a) Plainly, \( 2 = (\sqrt{2})^2 \). Explain why \( 2 \) is the only prime which ramifies in \( \mathbb{Z}[\sqrt{2}] \).

   (b) Let \( p \) an odd prime. Then, show that \( 2 \) a square modulo \( p \) iff \( p \equiv \pm1 \mod 8 \). You may use the fact that \( \left( \frac{2}{p} \right) = (-1)^{(p^2-1)/8} \); remember we want \( \left( \frac{2}{p} \right) = 1 \).

   (c) From part (b), describe which rational primes remain prime in \( \mathbb{Z}[\sqrt{2}] \). What happens otherwise?

   (d) Illustrate your findings by factoring the positive rational primes less than 50.

3. Show that the class number for the number rings of \( \mathbb{Q}(\sqrt{d}) \) is equal to 1 for \( d = -19, -43, -67, -163 \). (This finishes the proof that the imaginary quadratic number rings with \( d = -1, -2, -3, -7, -11, -19, -43, -67, -163 \) all are UFDs.)

4. In the number ring \( \mathbb{Z}[\sqrt{-6}] \):

   (a) Show that every ideal is equivalent to one of norm at most 3.

   (b) Verify that \( \langle 2 \rangle = \langle 2, \sqrt{-6} \rangle^2 \) and \( \langle 3 \rangle = \langle 3, \sqrt{-6} \rangle^2 \). Conclude that the only ideals of norm 2, 3 are \( \langle 2, \sqrt{-6} \rangle \) and \( \langle 3, \sqrt{-6} \rangle \). (Hence \( h \leq 3 \).)

   (c) Using \( \langle 2, \sqrt{-6} \rangle^2 = \langle 2 \rangle \) or otherwise, show that \( h = 2 \).

   (d) Find principal ideals \( a, b \) in \( \mathbb{Z}[\sqrt{-6}] \) such that \( a\langle 2, \sqrt{-6} \rangle = b\langle 3, \sqrt{-6} \rangle \).

5. If \( K \) is a number field of degree \( n \), prove that

\[
|\Delta| \geq \left( \frac{\pi}{4} \right)^n \left( \frac{n^n}{n!} \right)^2.
\]